Complexity of branch-and-bound and cutting planes in mixed-integer optimization

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Joint work with
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Goals and motivation

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- Show that Branch-and-Cut (BC) can be "exponentially better" than BB and CP alone

Motivation:

- BB and CP are (among) the main general-purpose techniques for mixed-integer optimization, but little is known on their relative strength
- Computationally, BC tends to be far more efficient and effective than BB and CP alone

The setting

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & x \in C \\ & x \in S \end{aligned}$$

where

- $C \subseteq \mathbb{R}^n$ is a closed convex set
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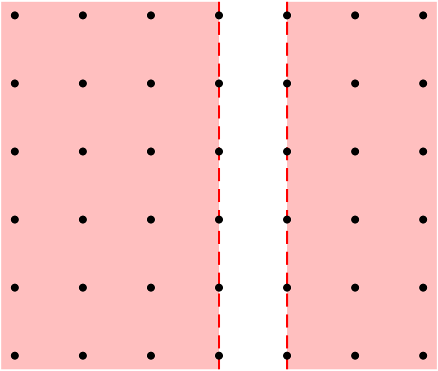
Typical case: (mixed) integer linear programming:

$$\begin{aligned} \min \ & c^\top x \\ \text{s.t. } & Ax \leq b \\ & x \in \mathbb{Z}^n \end{aligned}$$

- C is a polyhedron $\{x \in \mathbb{R}^n : Ax \leq b\}$
- $S = \mathbb{Z}^n$

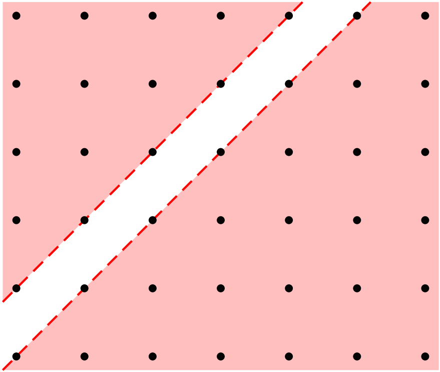
Disjunctions

Variable disjunction: $D = \{x : x_i \leq b \text{ or } x_i \geq b+1\}$, where $b \in \mathbb{Z}$



Disjunctions

Split disjunction: $D = \{x : a^\mathsf{T} x \le b \text{ or } a^\mathsf{T} x \ge b+1\}$, where $a \in \mathbb{Z}^n$ and $b \in \mathbb{Z}$.



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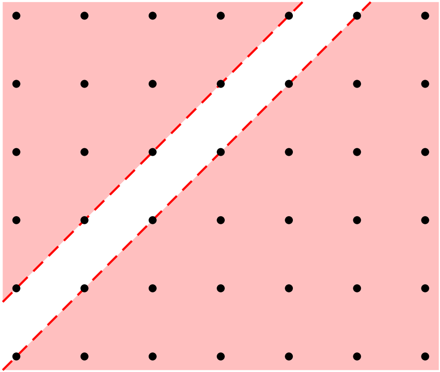
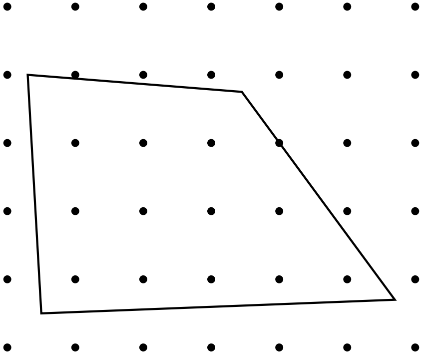


General disjunction: a finite union of polyhedra that cover S.

Branch-and-bound



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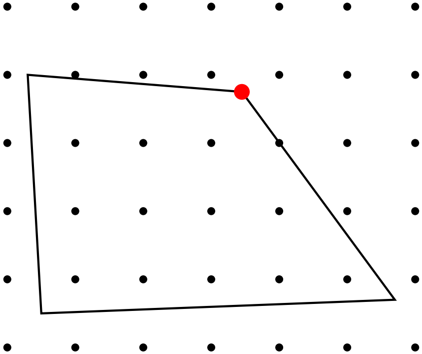

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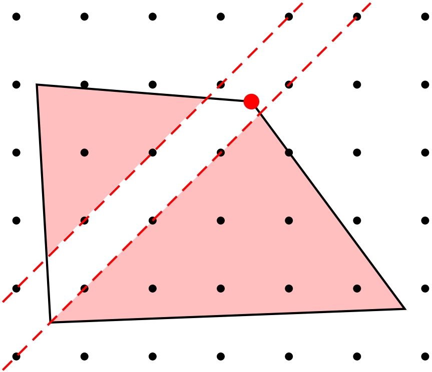

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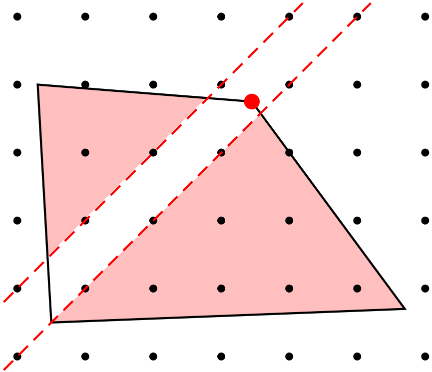
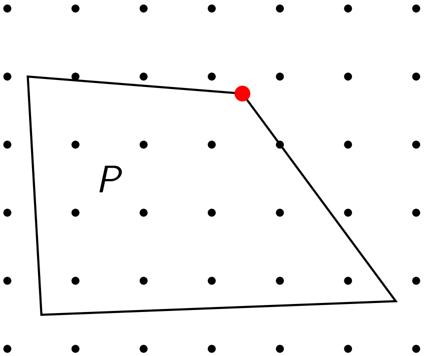


We assume that the best-node strategy is used: then the first feasible solution found is optimal.

Cutting planes



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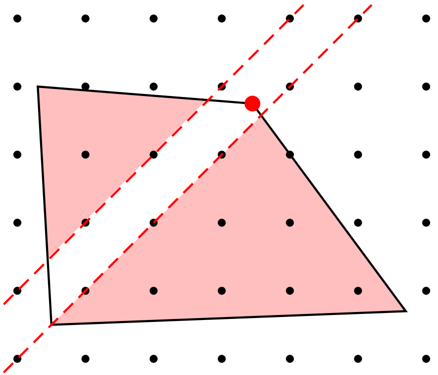

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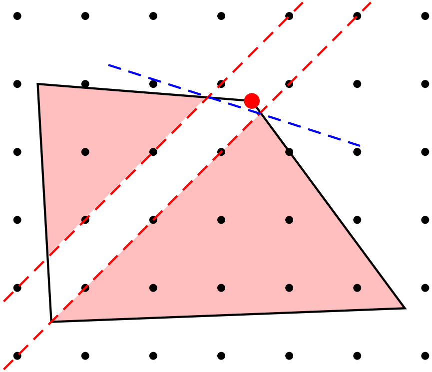

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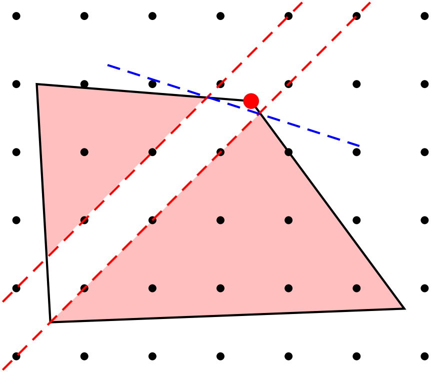

Disjunctive cut: any linear inequality valid for $P \cap D$, where D is a disjunction (split cut if D is a split disjunction).

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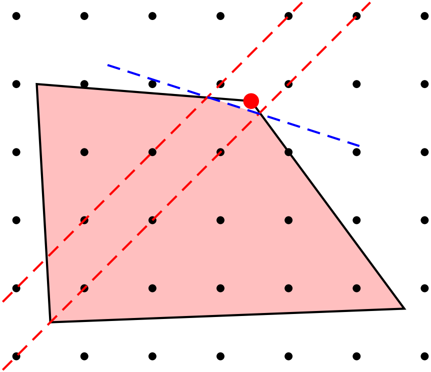

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We compare the number of nodes (length) produced by these algorithms based on the same families of disjunctions, assuming optimal choices.

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Proofs are stronger than algorithms, even in dimension 2 (Owen & Mehrotra 2001).

Summary of comparison between BB and CP

	variable disjunctions		split disjunctions	
	0/1 sets	general sets	0/1 sets	general sets
Variable dim.				
Fixed dim.				

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Variable dim.				
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Variable dim.	CP $\leq$ BB			
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0/1 convex sets, variable disjunctions

Theorem (Dash 2003/Chvátal 1973)

Let $P \subseteq [0,1]^n$ be a polytope. If a valid inequality for $P \cap \mathbb{Z}^n$ has a BC proof/algorithm of length N based on variable disjunctions, then it has a CP proof/algorithm of length N based on variable disjunctions.

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Theorem (BCDJ 2022)

Let $C \subseteq [0,1]^n$ be a closed convex set. If a valid inequality $cx \leq \gamma$ for $C \cap \mathbb{Z}^n$ has a BC proof/algorithm of length N based on variable disjunctions, then $cx \leq \gamma + \epsilon$ has a CP proof/algorithm of length N based on variable disjunctions, for any $\epsilon > 0$.

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Question

Can ϵ be removed?

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Summary of comparison between BB and CP

	variable disjunctions		split disjunctions	
	0/1 sets	general sets	0/1 sets	general sets
Variable dim.	CP $\leq$ BB CP *poly*(n) vs BB *exp*(n)			
Fixed dim.	BB $O(1)$ CP $O(1)$		BB $O(1)$ CP $O(1)$	

Exponential-gap instances

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(This example can be made less pathological.)

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General split disjunctions

Theorem (BCDJ 2022)

Let C be a closed convex set. If a valid inequality for C has a CP proof of length N based on general split disjunctions, then it has a BB proof of length $3N$ based on general split disjunctions.

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This can be extended to arbitrary disjunctions, provided that all split disjunctions are included.

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Polynomial-gap instances

Theorem (Conforti, Del Pia, DS, Faenza, Grappe 2015)

For general polytopes and general split disjunctions in fixed dimension, there are examples in which BB takes in $O(1)$ iterations while CP needs $poly(data)$ iterations.

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Is the split rank polynomial in variable/fixed dimension?

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Fixed dim.	BB $O(1)$ CP $O(1)$	BB *poly*(CP) BB $O(1)$ vs CP ∞	BB $O(1)$ CP $O(1)$	BB $\leq 3{\cdot}$CP BB $O(1)$ vs CP *poly*(*data*)

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Fixed dim.	BB $O(1)$ CP $O(1)$	BB $poly$(CP) BB $O(1)$ vs CP ∞	BB $O(1)$ CP $O(1)$	BB $\leq 3 \cdot$CP BB $O(1)$ vs CP $poly(data)$

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A branching scheme based on a family of disjunction $\mathcal{D}$ and a CP paradigm are complementary if there is a family of instances where CP gives polynomial size proofs and the shortest BB proof based on $\mathcal{D}$ is exponential, and there is another family where the opposite happens.

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Theorem (BCDJ 2022; here informal)

Under the above complementarity assumption, there are instances where BC does exponentially better than BB and CP alone.

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- Q1: Is BC superior to BB and CP alone precisely when BB and CP are complementary?
- ILP is polynomial in fixed dimension (Lenstra 1983) and particularly fast in dimension 2 (Eisenbrand, Laue 2005).
  Q2: Is there a cutting plane algorithm (perhaps based on split disjunctions) that solves ILP in polynomial time in fixed dimension?
  (This is true in dimension 2 (BCDJ 2021).)

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