Quantum algorithms in “modern” combinatorial optimization

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"Modern" combinatorial optimization

combine tools from

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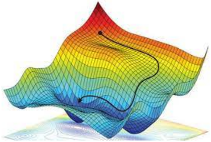
combine tools from

continuous optimization

- interior point methods
- (accelerated) gradient descent
- second-order methods



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continuous optimization and discrete optimization

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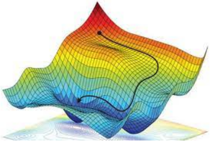
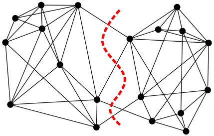


- graph sparsification
- combinatorial preconditioning
- random walks/electric networks



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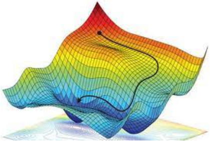

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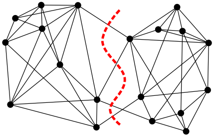

e.g., best maximum flow algorithms

use IPMs…
with fast Laplacian solvers …

…based on graph sparsifiers

This talk

quantum algorithms catch up:

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3) matrix scaling and second-order methods (‘10-‘20)

Quantum model

quantum time complexity

= total number of elementary gates, queries to input*,
QRAM operations

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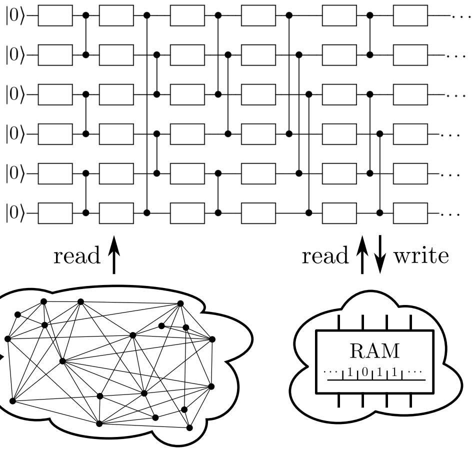

*typically, adjacency matrix of input graph

Quantum primitives

key routine = Grover search

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consider N elements, k of which are “marked”
Grover search finds marked element in time $O(\sqrt{N/k})$

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amplitude amplification, amplitude estimation
generalize to "marked" subspaces

Disclaimer

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- new insights in classical algorithms
  (similar to streaming, dynamic, distributed, . . . settings)

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- (subquadratic) polynomial speedup
- QRAM requirements

Motivation:

- long-term applications, better understanding of quantum computing
- new insights in classical algorithms
  (similar to streaming, dynamic, distributed, . . . settings)
- “true” complexity of a problem?
  (e.g., complexity of matrix multiplication)

Quantum algorithms for

1) cut sparsification and cut problems (‘90-‘00)

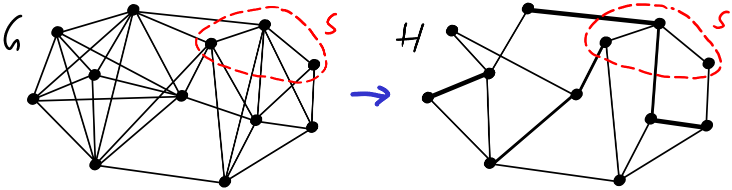
2) spectral sparsification and Laplacian solving (‘00-‘10)

3) matrix scaling and second-order methods (‘10-‘20)

Cut sparsification

ϵ-cut sparsifier H of G is sparse subgraph such that cuts in H approximate cuts in G:

$$\mathrm{val}_H(S) = \sum_{x \in S, y \notin S} w_H(x,y) = (1 \pm \epsilon)\, \mathrm{val}_G(S), \quad \forall S \subset V.$$



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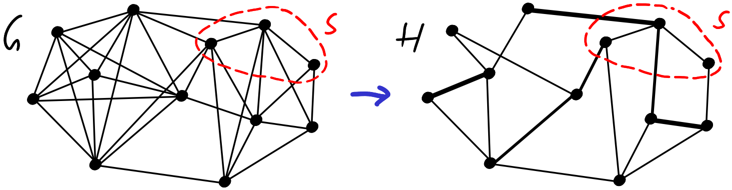
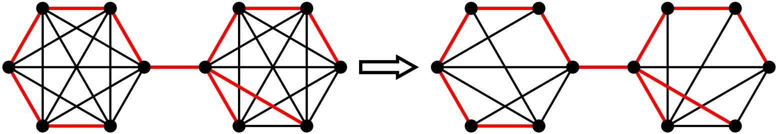


[Benczúr-Karger '96]: exists ϵ-cut sparsifier with $\widetilde{O}(n/\epsilon^2)$ edges

= building block of many classical algorithms for graph problems

Algorithm for cut sparsification



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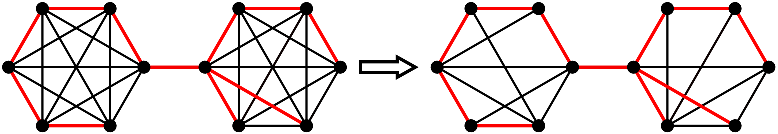

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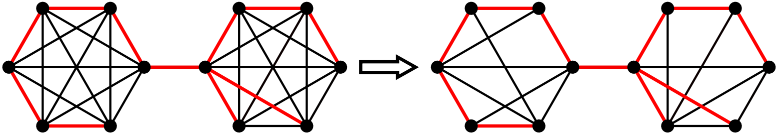

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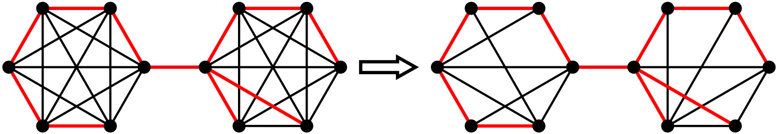

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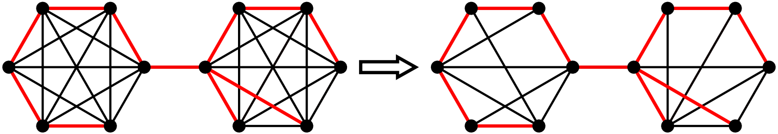

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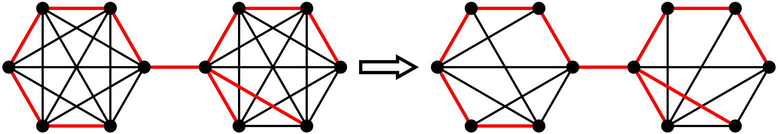

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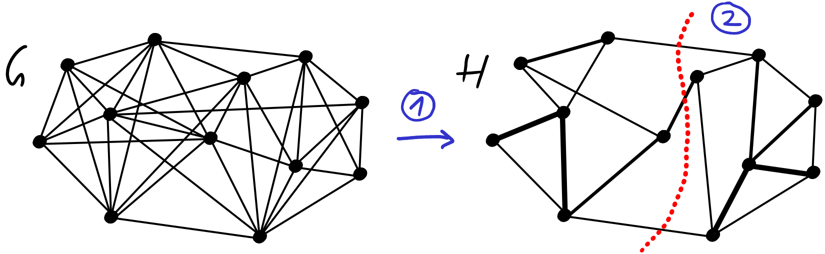
using quantum algorithm for MST* + more work:

Theorem (A-de Wolf '20)

There is a quantum algorithm for constructing an ϵ-cut sparsifier H in time $\widetilde{O}(n^{3/2}/\epsilon)$, which is optimal.

*[Dürr-Heiligman-Høyer-Mhalla '08]

Quantum speedup for cut problems



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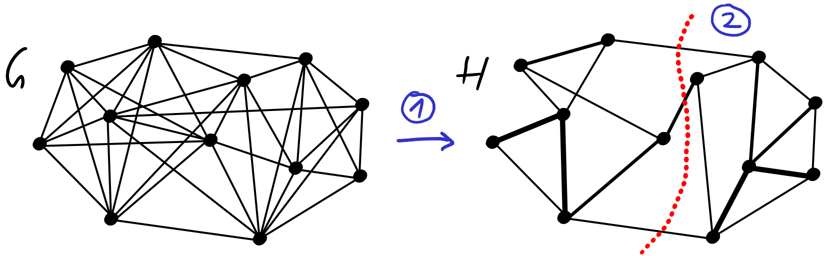

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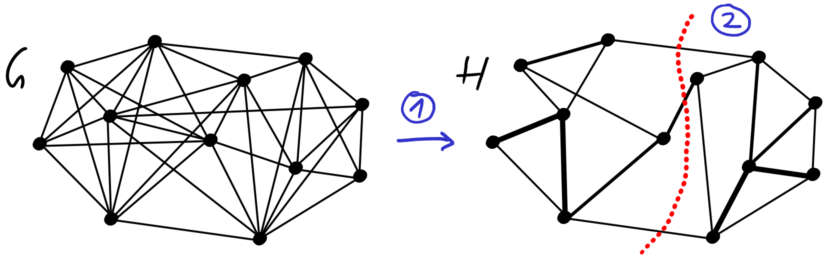

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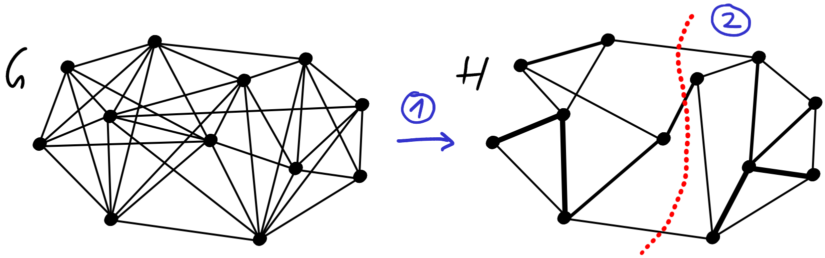

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= approximate minimum cut, maximum cut, sparsest cut, . . . in time

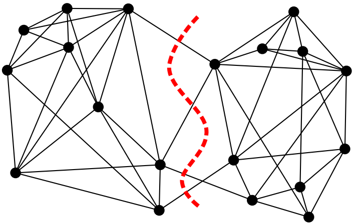
$$\widetilde{O}(n^{3/2}/\epsilon)$$

versus $\Omega(n^2)$ classically

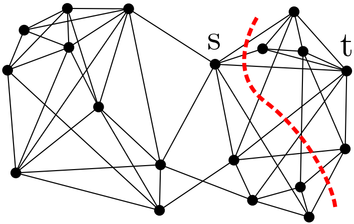
Quantum speedup for cut problems

With more work (on simple graphs):

exact minimum cut
in time $\widetilde{O}(n^{3/2})$, optimal
[Apers-Lee '20]



exact minimum s-t cut
in time $\widetilde{O}(n^{11/6})$, suboptimal?
[Apers-Auza-Lee '21]



versus $\Omega(n^2)$ classically

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spectral sparsifier H is sparse subgraph of G such that "quadratic forms" in Laplacian L_H approximate those in L_G:

$$x^T L_H x = \sum_{i<j} w_H(i,j)(x_i - x_j)^2 = (1 \pm \epsilon) x^T L_G x, \quad \forall x \in \mathbb{R}^n$$

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hence, spectral sparsifier $\Rightarrow$ cut sparsifier

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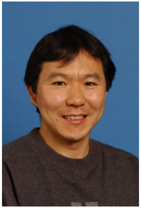

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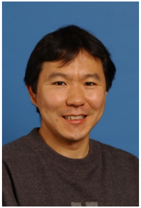

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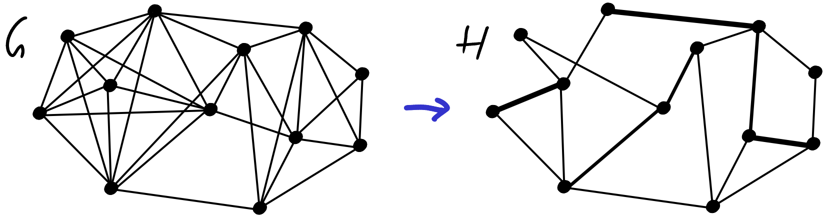
Quantum speedup for Laplacian solving



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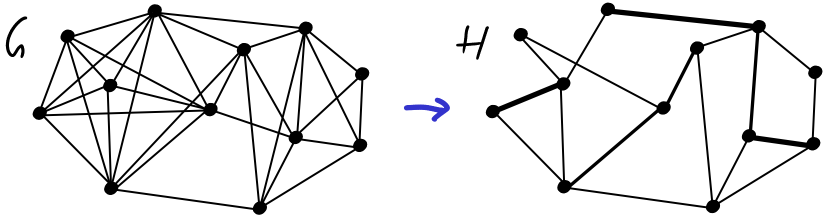

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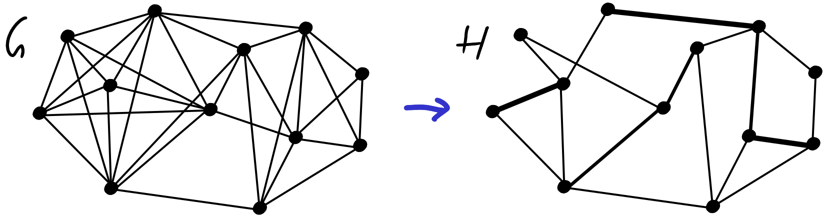

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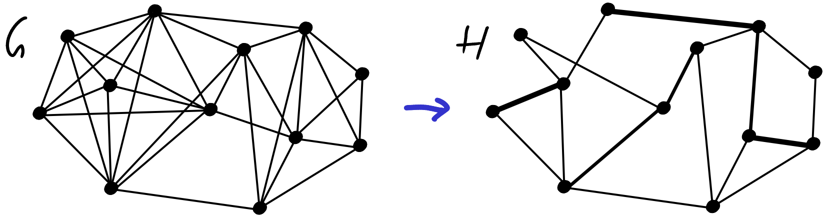

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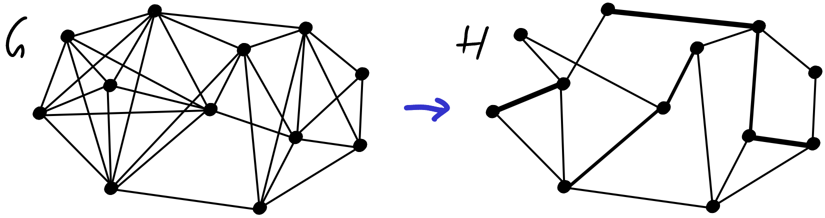

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similarly, quantum speedups for spectral clustering, RW properties, ...

Quantum algorithms for

1) cut sparsification and cut problems (‘90-‘00)

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3) matrix scaling and second-order methods (‘10-‘20)

Matrix scaling

Input:

matrix $A \in \mathbb{R}^{n\times n}_{>0}$, target marginals $r, c \in \mathbb{R}^n_{>0}$

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Goal:

find "rescaling"

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such that XAY has row sums r and column sums c

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Applications:

approximating matrix permanent, optimal transport in machine learning, numerical linear algebra, ...

Algorithms for matrix scaling

"folklore" classical algorithm:

complexity $\widetilde{O}(n^2/\epsilon)$ for ϵ-approximate solution
by iterative rescaling

[Sinkhorn '64]

$$A \to \underbrace{X_1 A}_{\text{fix } r} \to \underbrace{X_1 A Y_1}_{\text{fix } c} \to \underbrace{X_2 X_1 A Y_1}_{\text{fix } r} \to \dots$$

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$$\downarrow$$

quantum algorithm:

complexity $\widetilde{O}(n^{3/2}/\epsilon^3)$ for ϵ-approximate solution
by Sinkhorn + quantum approximate counting
[van Apeldoorn-Gribling-Li-Nieuwboer-Walter-de Wolf '21]

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observation:

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$\nabla f(x,y) = 0 \;\Leftrightarrow\; (x,y)$ describes rescaling

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Sinkhorn, $\widetilde{O}(n^2/\epsilon)$: first-order method

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improved algorithm:

complexity $\widetilde{O}(n^2 \log(1/\epsilon))$ using *second-order* method

[Cohen-Mądry-Tsipras-Vladu '17]

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key tool: Hessian of f is Laplacian matrix

$\rightarrow$ can use efficient Laplacian solving!

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complexity $\widetilde{O}(n^2 \log(1/\epsilon))$ using *second-order* method
[Cohen-Mądry-Tsipras-Vladu ’17]

? improved quantum algorithm for matrix scaling ?

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+ quantum sparsification and Laplacian solving

also, no quantum speedup for $\epsilon \ll 1$

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Future directions:

1) quantum speedup in interior point methods?
flagship problem = maximum flow (‘20-. . .)

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Future directions:

1) quantum speedup in interior point methods?
flagship problem = maximum flow (‘20-...)

2) continuous $\leftrightarrow$ discrete trends also in sampling algorithms
e.g., logconcave sampling, estimating volume of convex bodies, ...