An introduction to the 1-2-3 Conjecture (and related problems)

Julien Bensmail
Université Côte d'Azur, France

42èmes Journées Franciliennes de Recherche Opérationnelle (JFRO)
Conjectures – Des réponses aux grandes questions sur la recherche opérationnelle, l'univers et le reste

Online event
September 13, 2021

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$1 + 3 =$

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Outline of the talk

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- **Families of variations:**
  - Playing with parameters to approach the conjecture
  - Generalisations to more general structures
  - Distinguishing at larger distance
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- **Conclusion and perspectives**

A disclaimer before we go

- **Main goal:** tell you a bit about the 1-2-3 Conjecture...
- ... and about the many open questions revolving around it
- $\Rightarrow$ Mostly about connections between the different problems

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- **Main goal:** tell you a bit about the 1-2-3 Conjecture...
- ... and about the many open questions revolving around it
- ⇒ Mostly about connections between the different problems
- ⇒ Not much overwhelming details, technicalities, etc.
  - all considered graphs are simple, loopless, undirected, connected
  - results obtained by numerous authors, since 2004
  - presented results do not follow chronological order
  - check the survey by Seamone (arXiv:1211.5122) for anything omitted

Introduction to the 1-2-3 Conjecture

The 1-2-3 Conjecture, in few words

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Edge weights and vertex colours

Michał Karoński and Tomasz Łuczak

Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Poznań, Poland
E-mail: karonski@amu.edu.pl and tomasz@amu.edu.pl

and

Andrew Thomason

DPMMS, Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WB, England
E-mail: a.g.thomason@dpmms.cam.ac.uk

Received 24th September 2002

Can the edges of any non-trivial graph be assigned weights from $\{1, 2, 3\}$ so that adjacent vertices have different sums of incident edge weights?

We give a positive answer when the graph is 3-colourable, or when a finite number of real weights is allowed.

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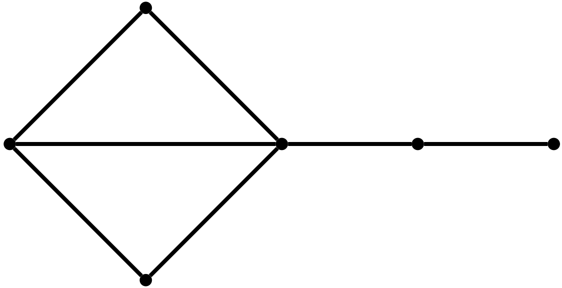
Terminology (may vary slightly along the talk):

- Labelling: labels $1, \dots, k$ assigned to the edges (for some $k \geq 1$)
- Colouring: colours (sums) of the vertices resulting from the labelling

Sample example



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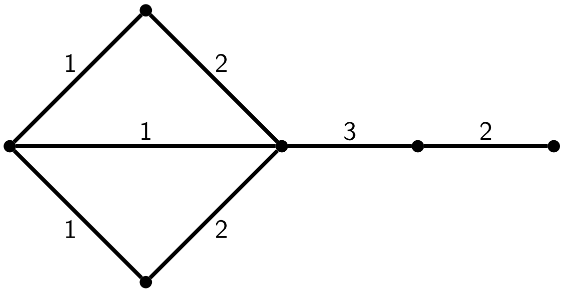

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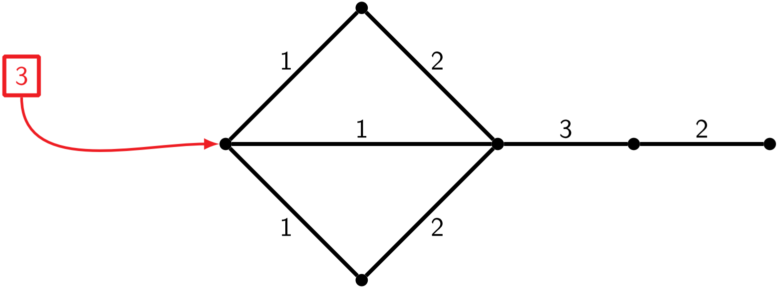

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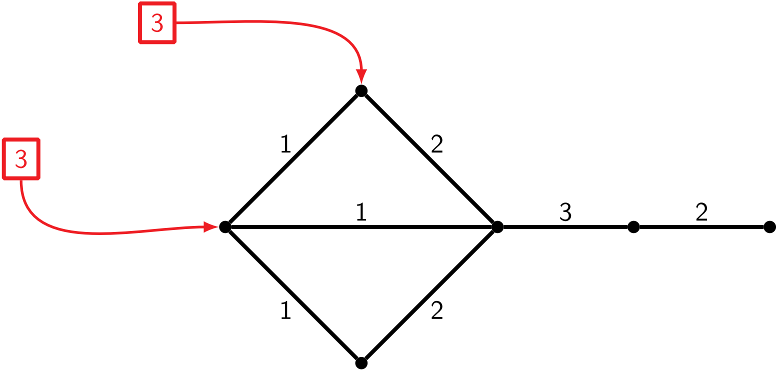

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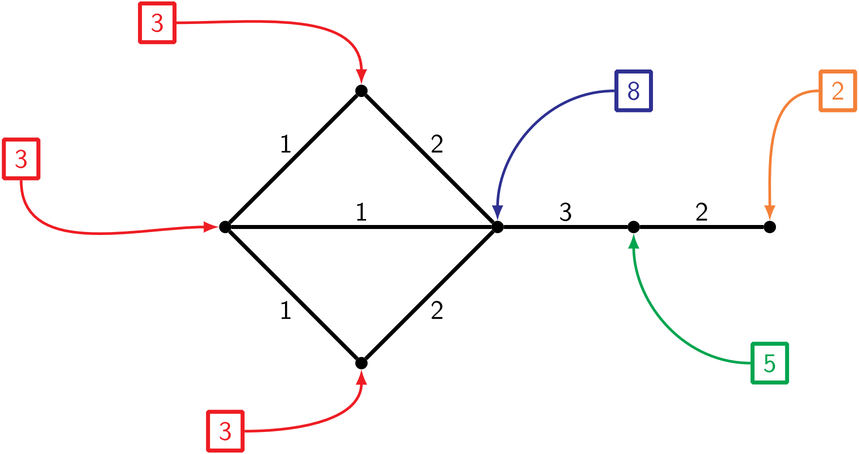

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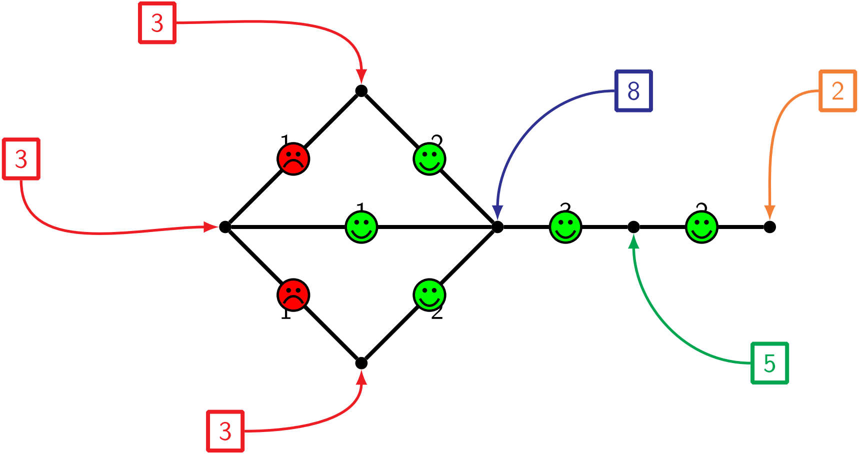
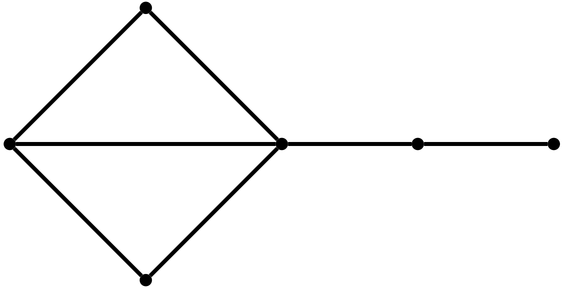


Sample example, 2nd try



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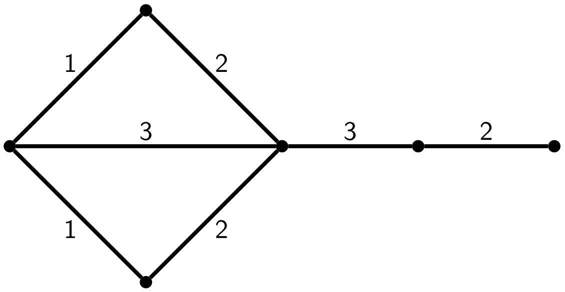

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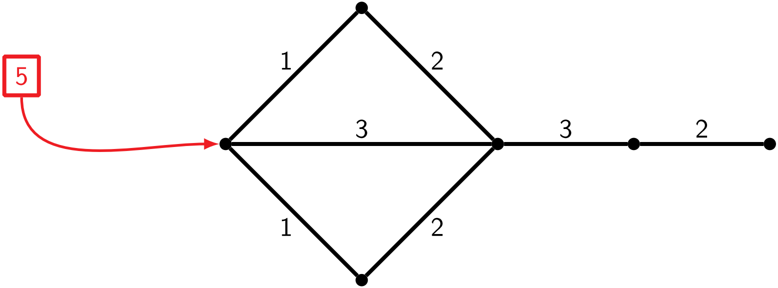

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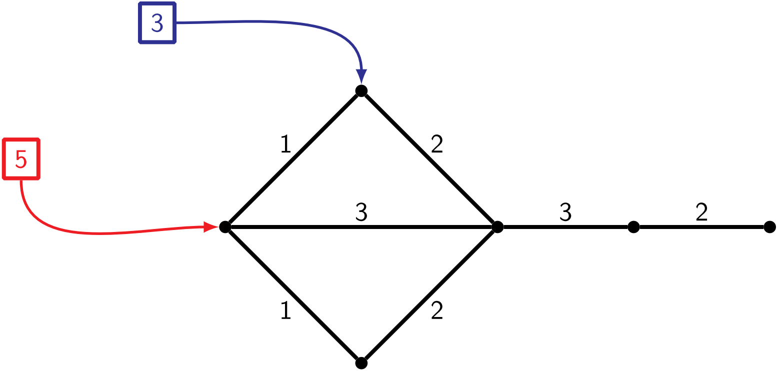

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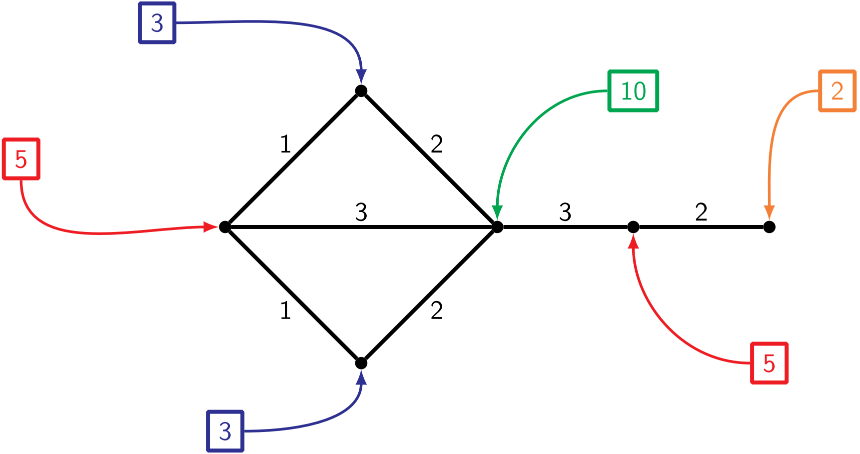

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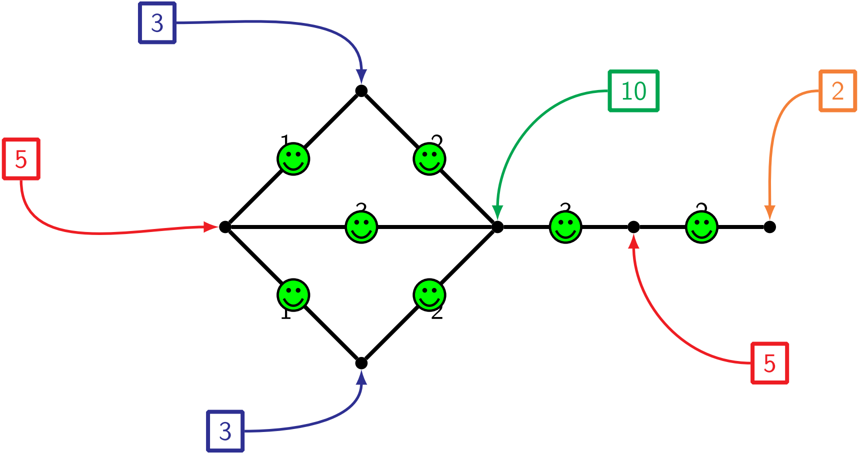

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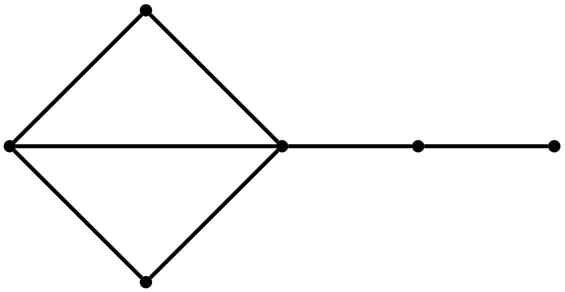

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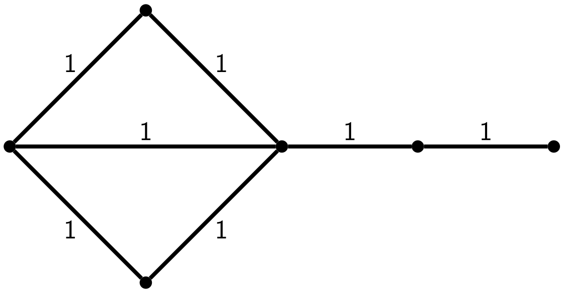

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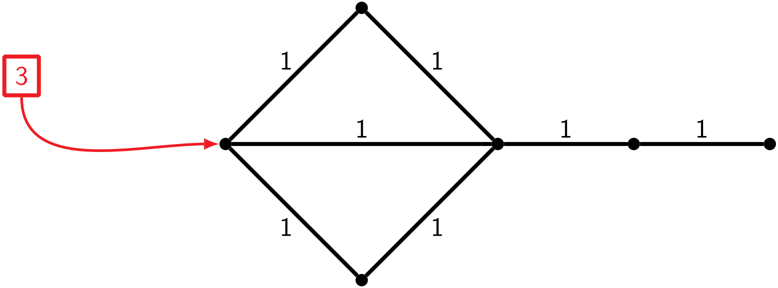

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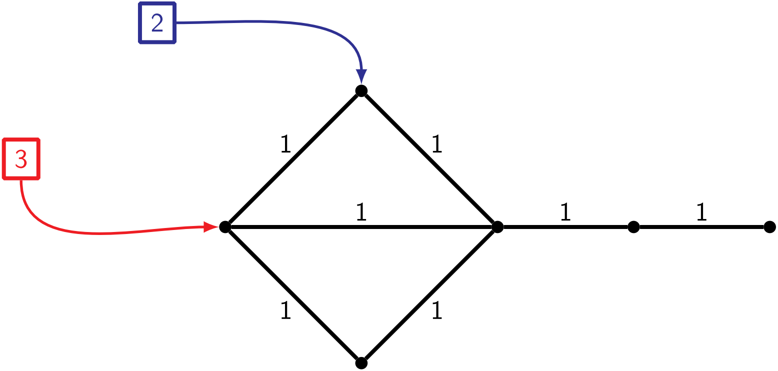

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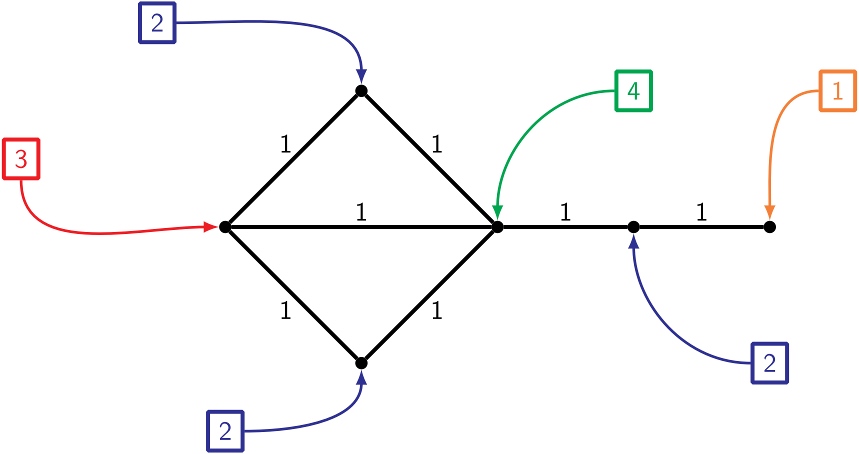

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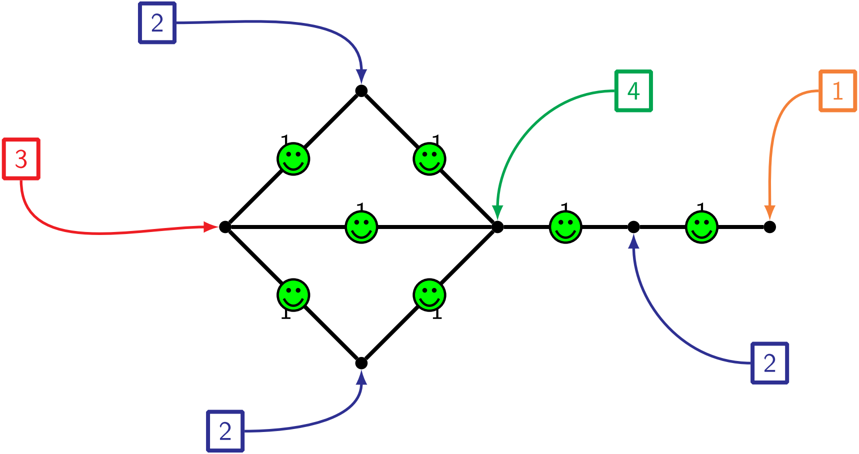

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This is always possible with $k \leq 3$.

- **But where does that come from?**
  - One of many **distinguishing labelling problems**
  - From the application p.o.v., vaguely related to **complex networks**
  - Related to **graph irregularity**, **proper vertex-colourings**, etc.
- But if you ask me, I would just suggest to see this all as a fun problem ☺

Two interpretations/motivations...

... leading to different questions

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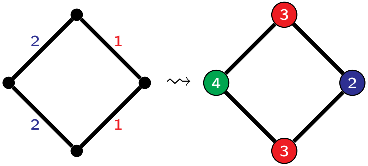

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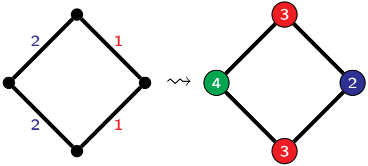

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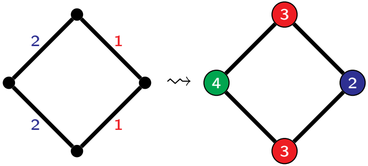

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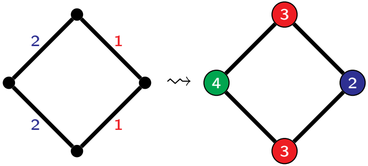

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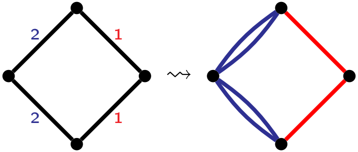

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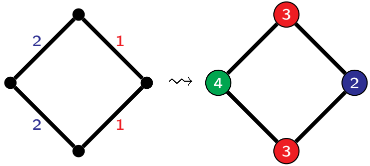

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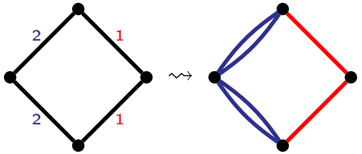

1-2-3 Conjecture

⇔

"Make" a graph locally irregular by multiplying every edge by at most 3

Most of what we know on the 1-2-3 Conjecture

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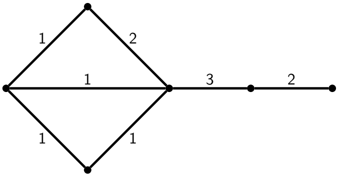
Also, many **side aspects, variants, etc.**, which are the topic of the talk ☺

– Families of variations –

Playing with parameters to approach the conjecture

Multiset version

- **Main difference:** vertex colour = **multiset** of incident labels



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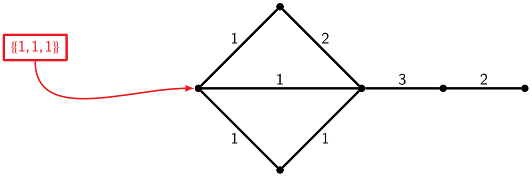

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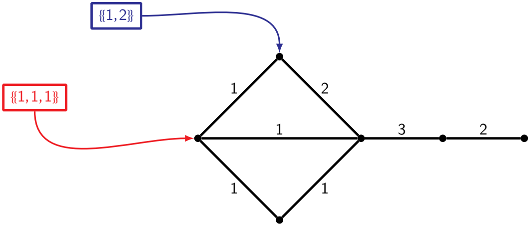

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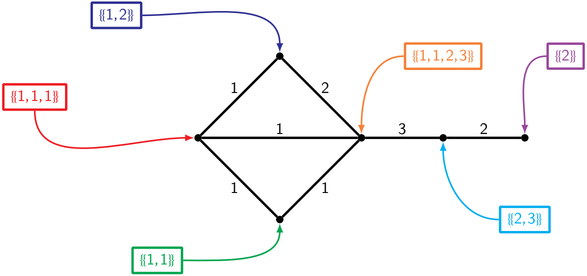

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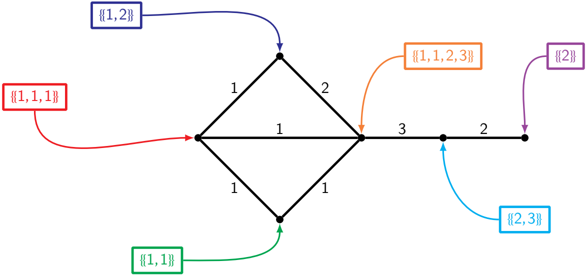

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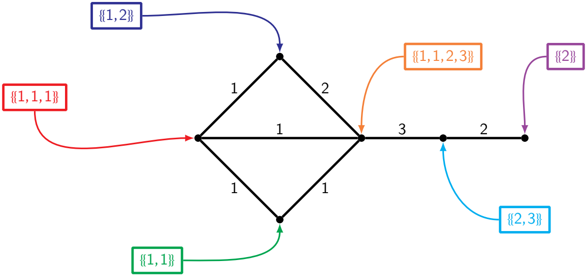

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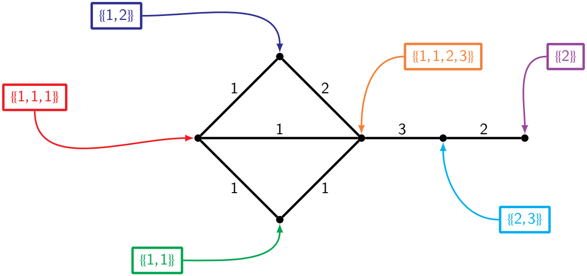

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  - Vučković (2018): **the conjecture is true!**

Product version

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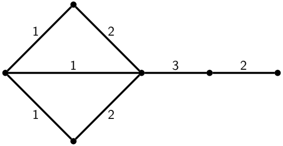

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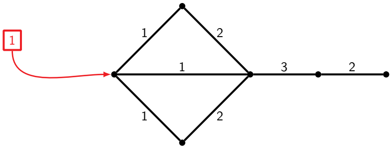

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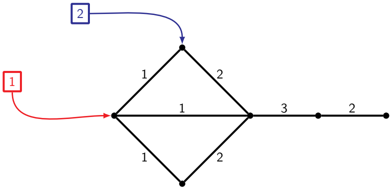

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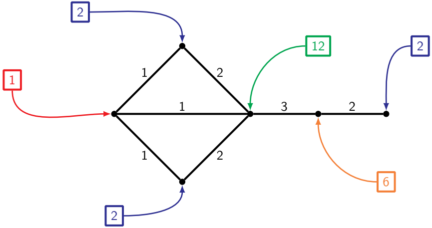

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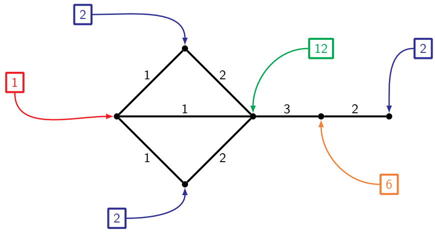

- **"In between" the multiset and sum versions:**
  - Distinct products ⇒ Distinct multisets
  - Distinct prime decompositions (into 2's and 3's) ⇒ Distinct products
  - 1's ~ ~~Deleting edge~~ Skipping labelling an edge

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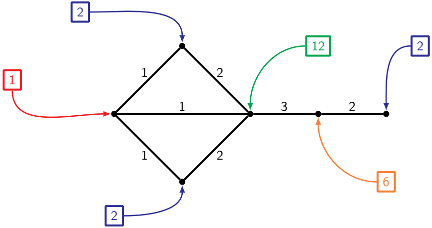

- **“In between” the multiset and sum versions:**
  - Distinct products ⇒ Distinct multisets
  - Distinct prime decompositions (into 2’s and 3’s) ⇒ Distinct products
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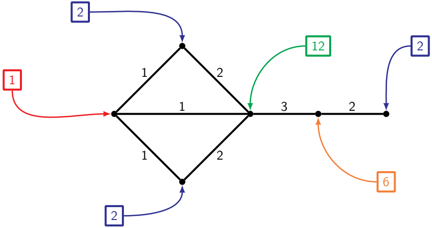

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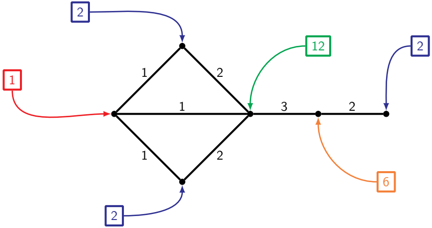

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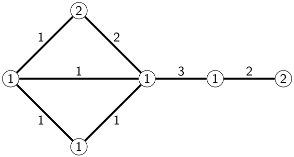

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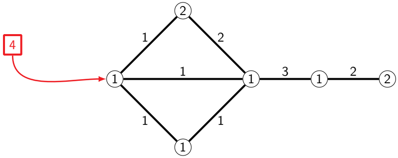

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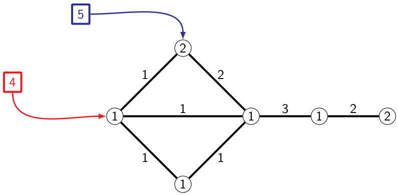

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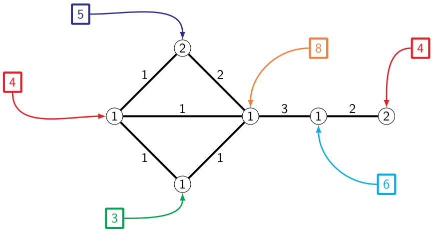

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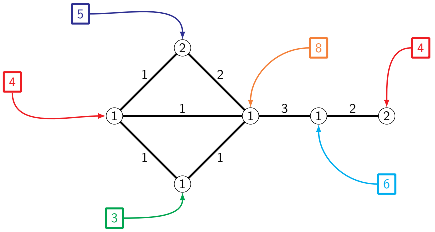

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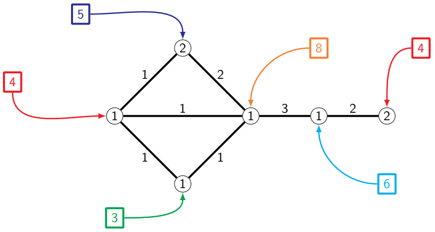

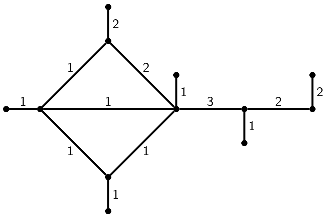

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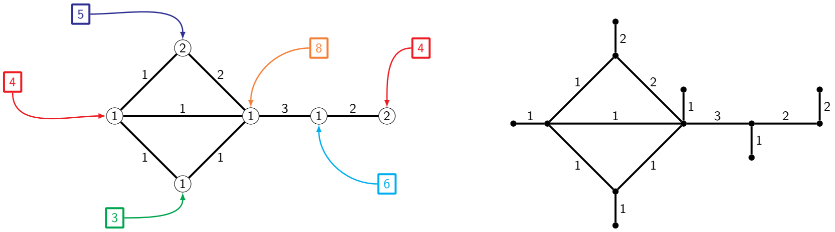

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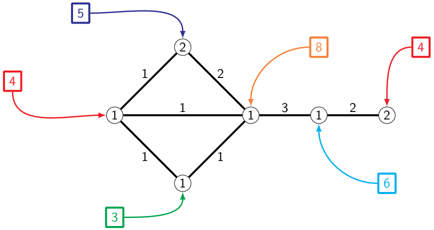

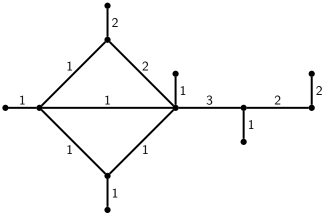

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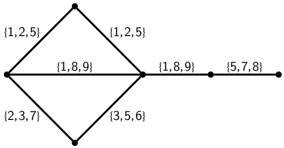
  - Kalkowski (2012?): 1,2,3 **on edges and** 1,2 **on vertices for all graphs**

List version

- **Main difference:** label edges with labels from **dedicated lists of size** ℓ



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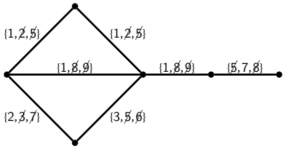

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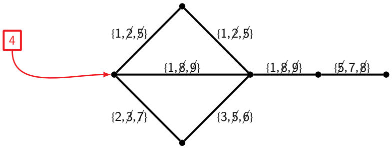

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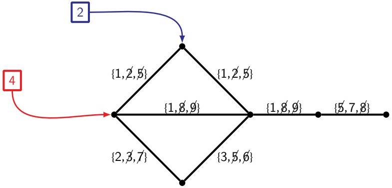

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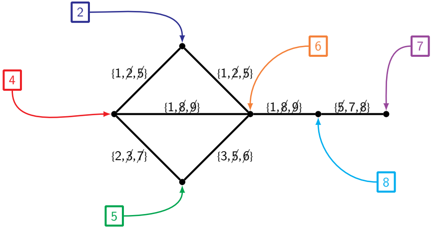

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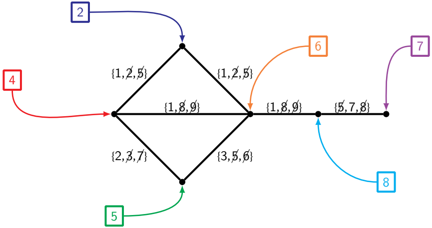

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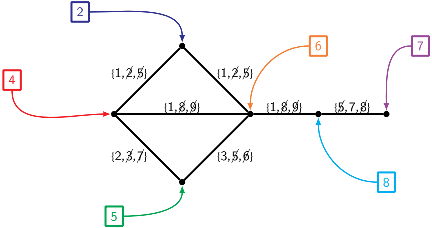

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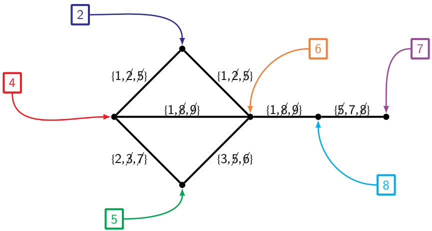

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  - Cao (2021): **Yes,** $\ell = 7$
  - Zhu (2021+): $\ell = 5$**!**

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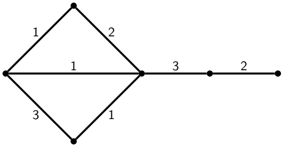

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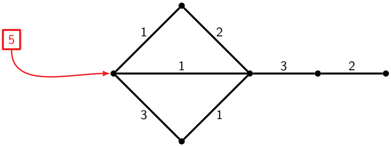

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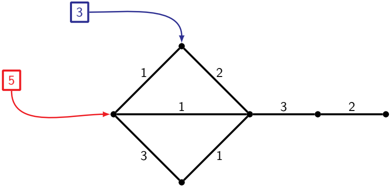

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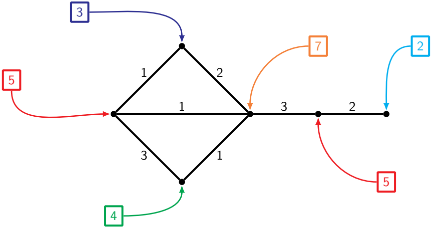

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Labels

# of 1's: 3
# of 2's: 2
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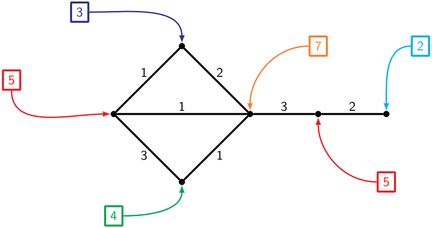

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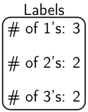

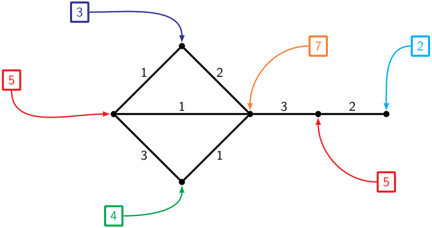

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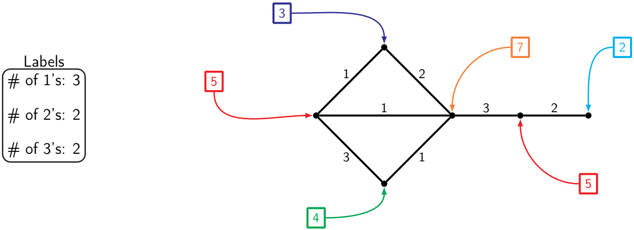

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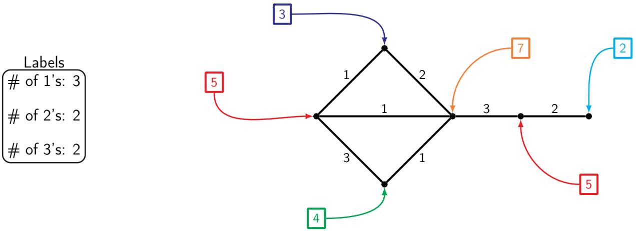

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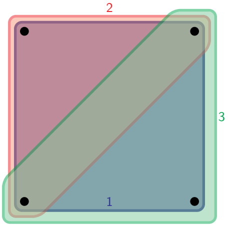
  - Also, infinitely many graphs that can be labelled with $1,2$ require $1,2,3$ here
  - **Still no constant number of labels $1,\dots,k$ is known to suffice**

– Families of variations –

Generalisations to more general structures

Hypergraphs

- **Main difference:** edge labels count for **all vertices in hyperedges**



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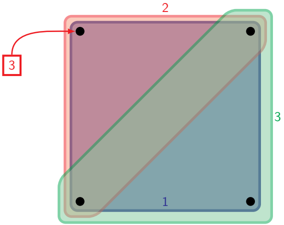

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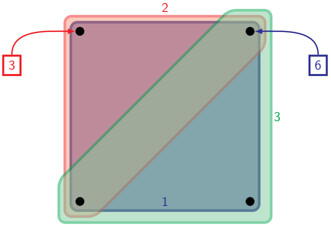

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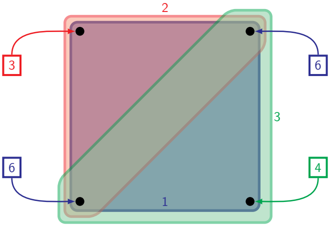

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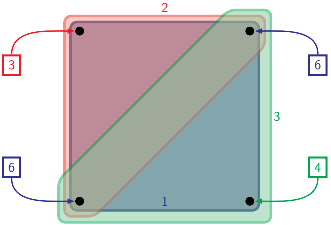

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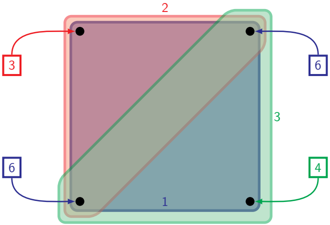

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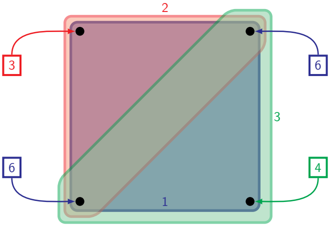

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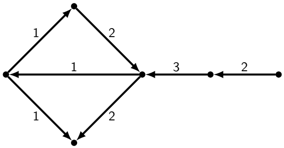
Directed graphs

- **Main difference:** two "types" of arcs ⇒ **many ways to compute colours**



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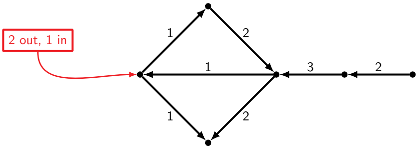

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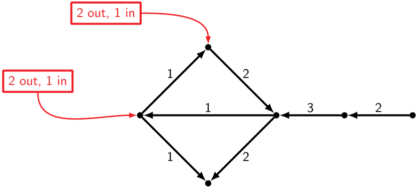

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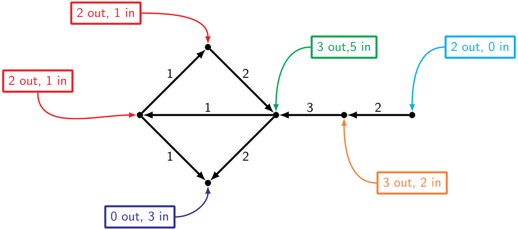

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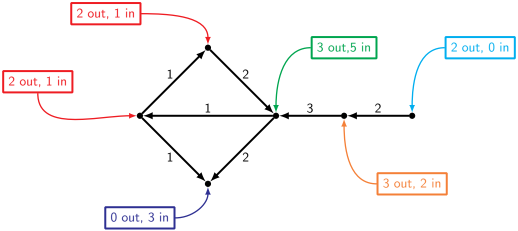

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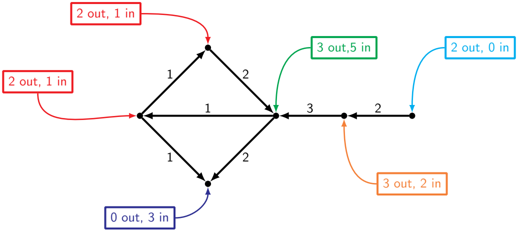

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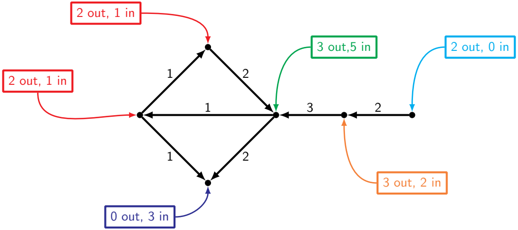

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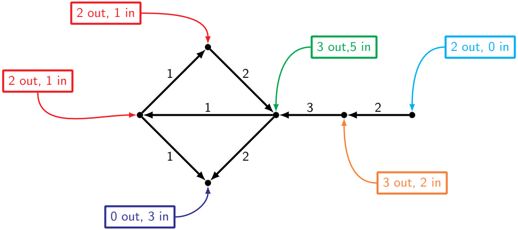

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  - $\text{in}(u) \neq \text{in}(v)$ (or $\text{out}(u) \neq \text{out}(v)$): 1,2,3 suffice (optimal), listing NP-hard
  - $\text{out}(u) \neq \text{in}(v)$: equivalent to the 1-2-3 Conjecture in bipartite graphs
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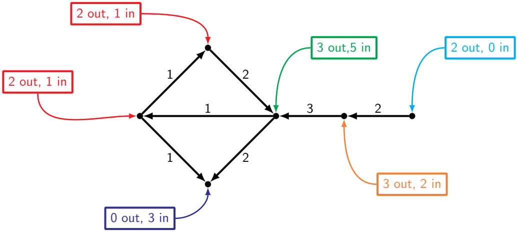
Directed graphs

- **Main difference:** two "types" of arcs ⇒ **many ways to compute colours**



- **Question:** A "challenging" variant, for some distinction condition?
  - Can all digraphs be labelled?
  - With a restricted set of labels?
  - Can we classify digraphs w.r.t. labels they need?
- **Some attempts (since 2012):**
  - ~~$\text{in}(u) + \text{out}(u) \neq \text{in}(v) + \text{out}(v)$: this is almost the 1-2-3 Conjecture ☺~~
  - $\text{in}(u) - \text{out}(u) \neq \text{in}(v) - \text{out}(v)$: 1,2 suffice (optimal)
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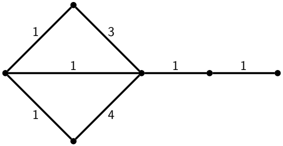
- **Most proofs are easy ☹**

– Families of variations –

Distinguishing at larger distance

Irregularity strength

- **Main difference:** must distinguish **all vertices** (not only adjacent ones)



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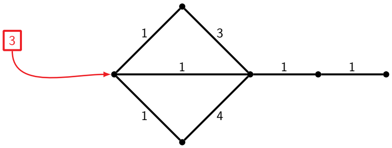

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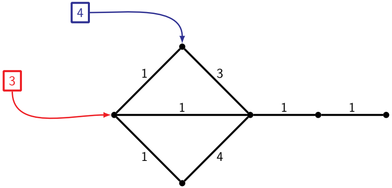

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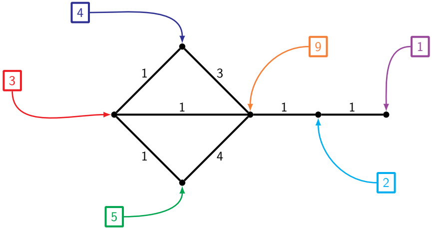

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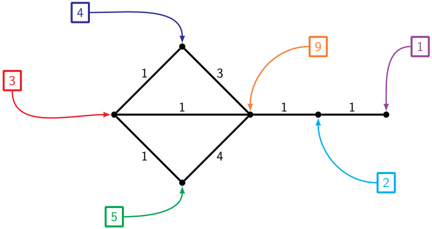

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  - Look for global irregularity
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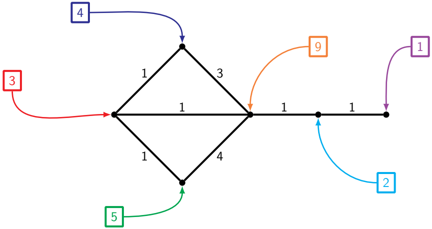

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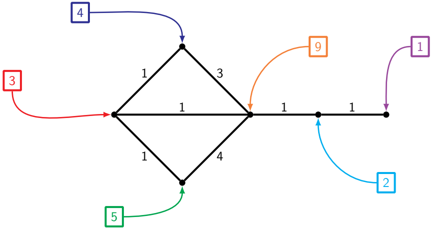

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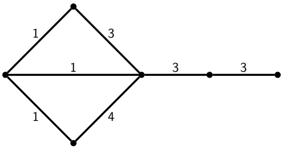
- **Conjecture (Aigner, Triesch, 1990):** $1, \dots, |V| - 1$ for all graphs?
  - Nierhoff (2000): **YES!**
  - Improved bounds in some cases
  - Kalkowski, Karoński, Pfender (2011): labels $1, \dots, \left\lceil \frac{6|V|}{\delta} \right\rceil$ suffice

Distant irregularity strength

- **Main difference:** distinguish only vertices **within a certain distance** r



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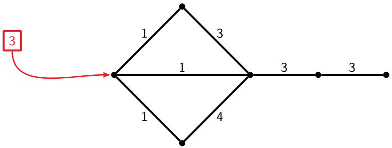

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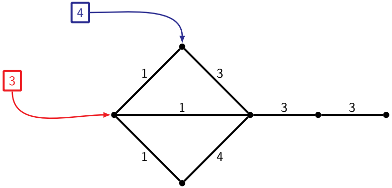

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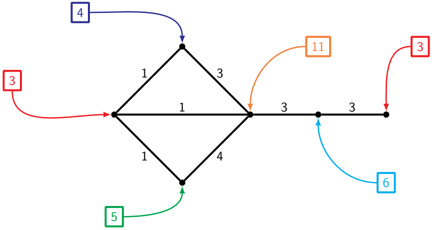

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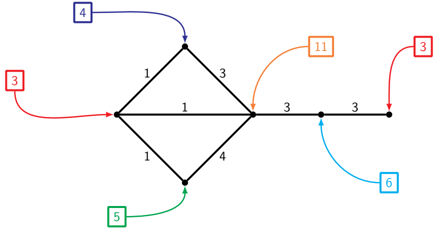

- **Irregularity strength with "limited distance":**
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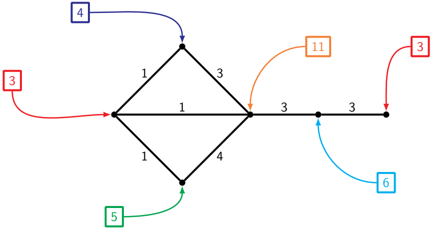

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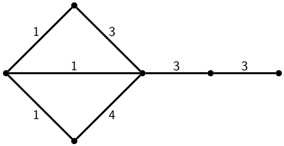
- **"Thread" (Przybyło, 2013):** smallest f_r s.t. $1, \ldots, f_r$ suffice for all graphs?
  - Przybyło proved that $f_r \leq 6\Delta^{r-1}$
  - Moore graphs show that $f_r \geq \Delta^{r-1}$
  - Improved in further works, sometimes for some graph classes

Wide version

- **Main difference:** also **colours** are fetched **at distance at most** r



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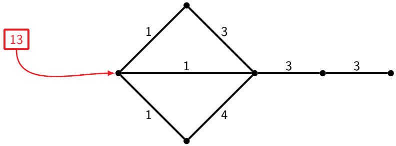

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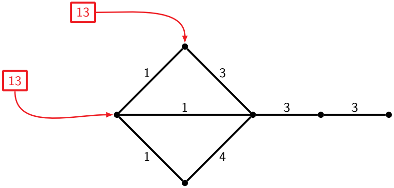

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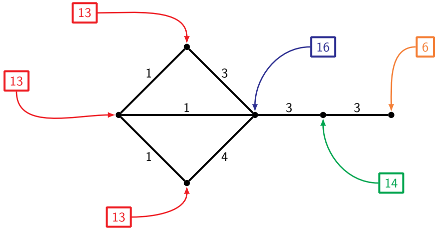

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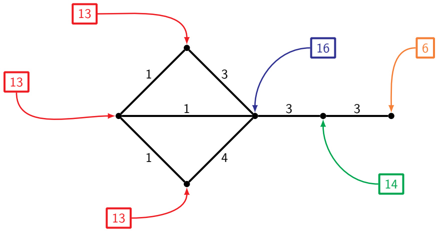

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  - $r = 1$: similar to the 1-2-3 Conjecture
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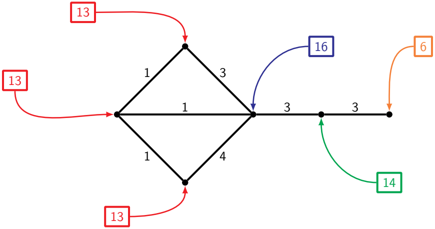

- **The two distance parameters coincide:**
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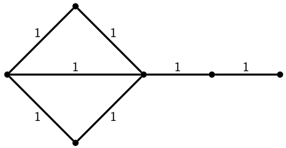
  - $r \geq 2$: quite different from the previous problems
  - Some connections with irregularity strength and the hypergraph version
- **"Thread" (B. *et al.*, 2021+):** smallest f_r s.t. $1, \ldots, f_r$ suffice for all graphs?
  - The authors proved that $f_r \leq \Delta^{2r-1}$
  - There are graphs showing that $f_r \geq 3 \cdot \Delta^{r-1}$
  - Nice phenomena (for instance, increasing $r \not\Rightarrow$ more labels)

– Families of variations –

Getting somewhat optimal

Minimising the number of distinct colours

- **Main difference:** number of distinct vertex colours **as small as possible**



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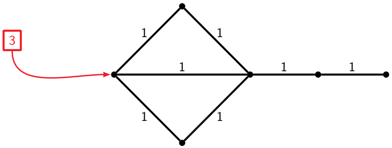

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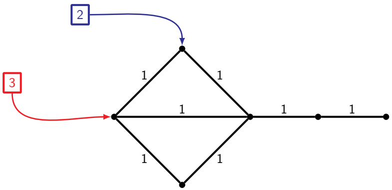

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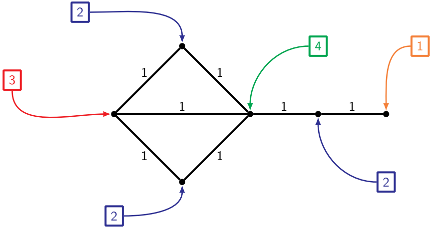

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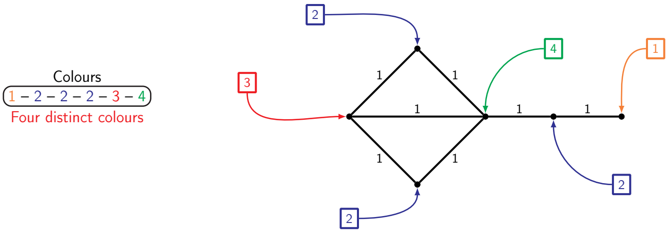

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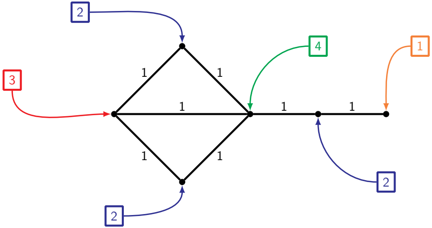
Colours
1 – 2 – 2 – 2 – 3 – 4
Four distinct colours



- **Get a “better” derived proper vertex-colouring/multigraph:**
  - Larger label is 3 $\not\Rightarrow$ Small number of distinct colours/vertex degrees
  - Can we get close to the chromatic number? With labels from any set?

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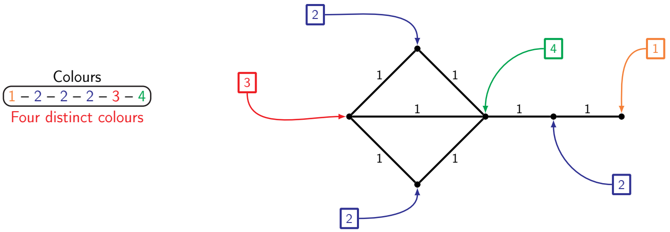

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- **Conjecture (B. *et al.*, 2019):** getting at most 2Δ distinct colours?
  - The 1-2-3 Conjecture, if true, would give at most 3Δ
  - Using relative numbers $\Rightarrow$ Close to the chromatic number
  - Bounds for some graph classes, for 1,2,3 (e.g. logarithmic bounds for trees)

Minimising the maximum colour

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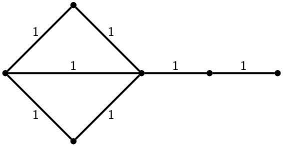

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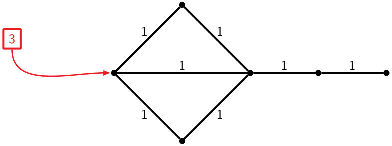

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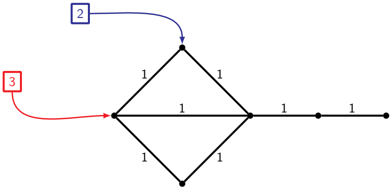

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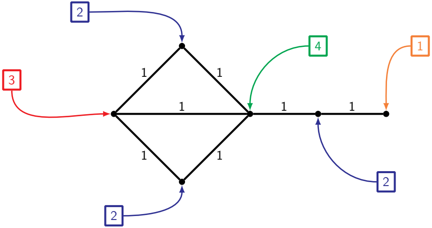

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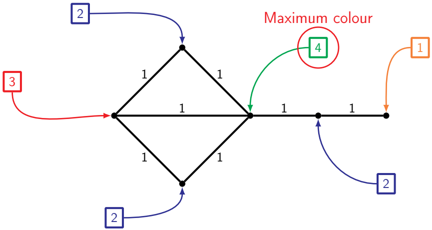

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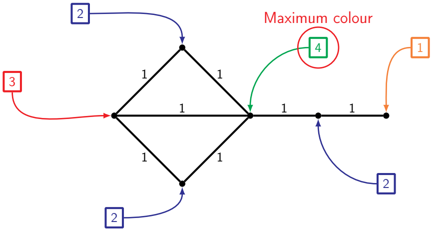

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  - **Might have to assign labels arbitrarily larger than** 3**!**

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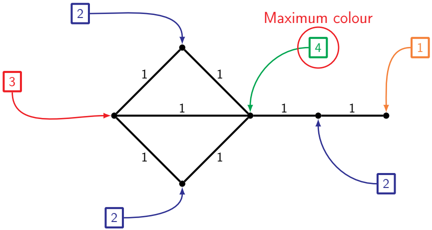

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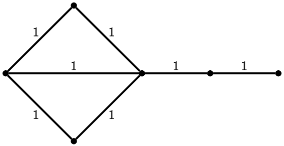
- **Conjecture (B. *et al.*, 2021):** getting maximum colour at most 2Δ?
  - The 1-2-3 Conjecture, if true, would give at most 3Δ
  - **The best result towards it yields 5Δ**
  - True for graphs that are complete, bipartite, etc.

Minimising the sum of assigned labels

- **Main difference:** assign labels **adding up to the smallest value possible**



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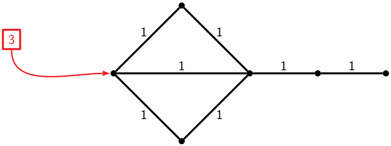

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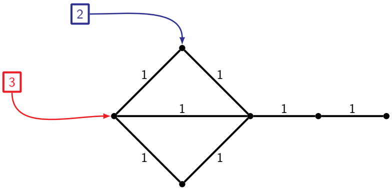

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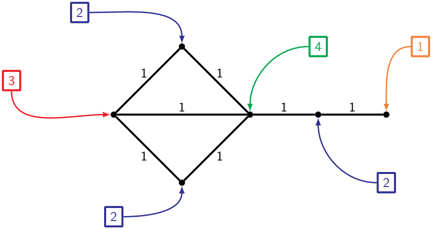

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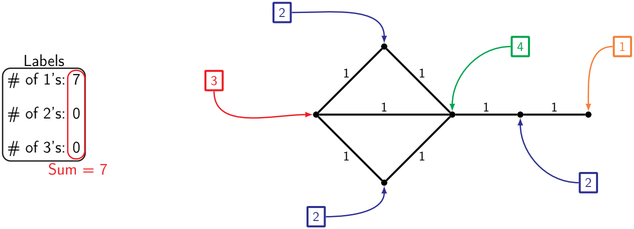

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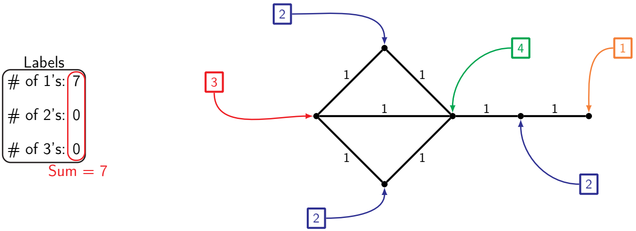

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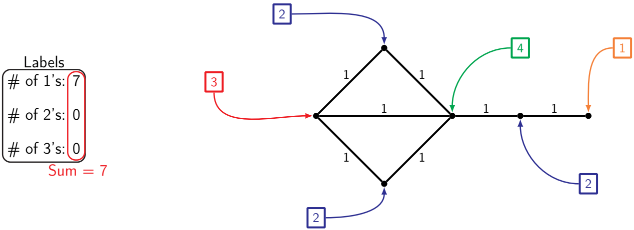

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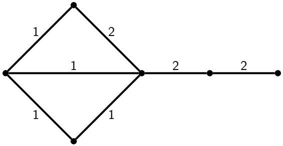
- **Conjecture (B. *et al.*, 2020):** assigning labels adding up to at most $2|E|$?
  - The 1-2-3 Conjecture, if true, would give at most $3|E|$
  - **The best result towards it yields** $5|E|$
  - Intuitively, approximately the same number of 1's and 2's, and "a few" 3's
  - True for graphs that are complete, bipartite, etc.

Minimising the number of 3's

- **Main difference:** assign label 3 to **as few edges as possible**



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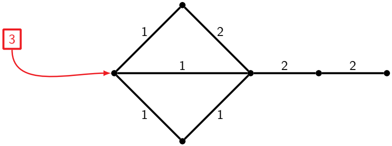

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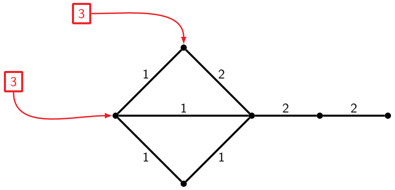

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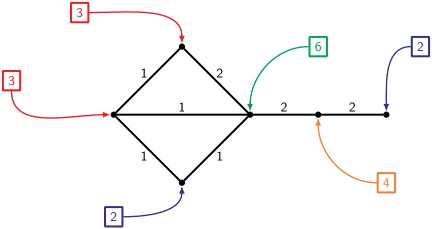

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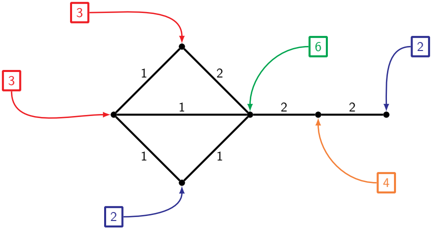
Labels

# of 1's: 4

# of 2's: 3

# of 3's: 0



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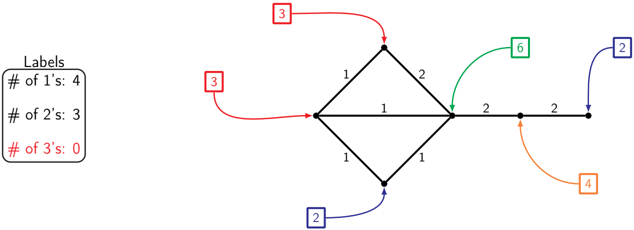
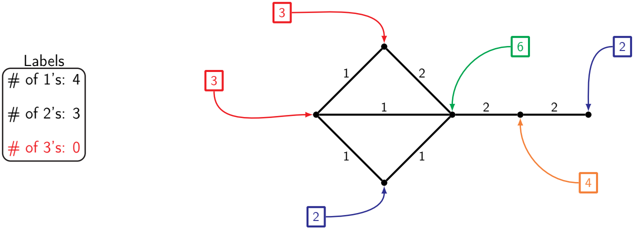


- **Is it true that 3's are barely needed?**
  - Presumption from the 1-2 Conjecture, the minimisation variants, etc.
  - "Tightness" of the 1-2-3 Conjecture

Minimising the number of 3's

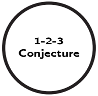
- **Main difference:** assign label 3 to **as few edges as possible**



- **Is it true that 3's are barely needed?**
  - Presumption from the 1-2 Conjecture, the minimisation variants, etc.
  - "Tightness" of the 1-2-3 Conjecture
- **Conjecture (B. *et al.*, 2021):** assigning 3 to at most $1/3$ edges?
  - Close to the conjecture for the equitable variant
  - True for graphs being bipartite, cubic, planar with large girth, cacti, etc.
  - Many 3-chromatic families require an unbounded number of 3's
  - **No general upper bound for all graphs**

A final picture

The 1-2-3 Conjecture universe



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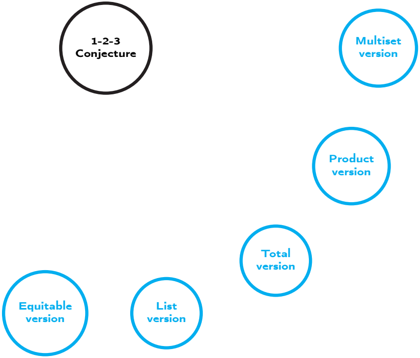

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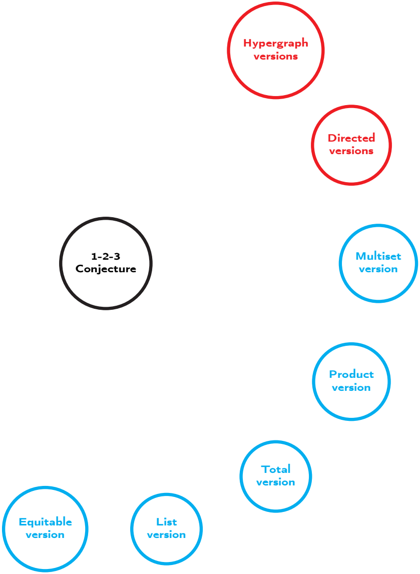

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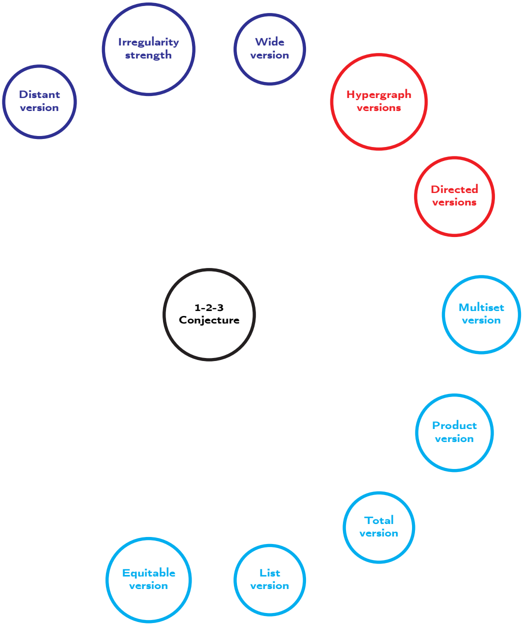

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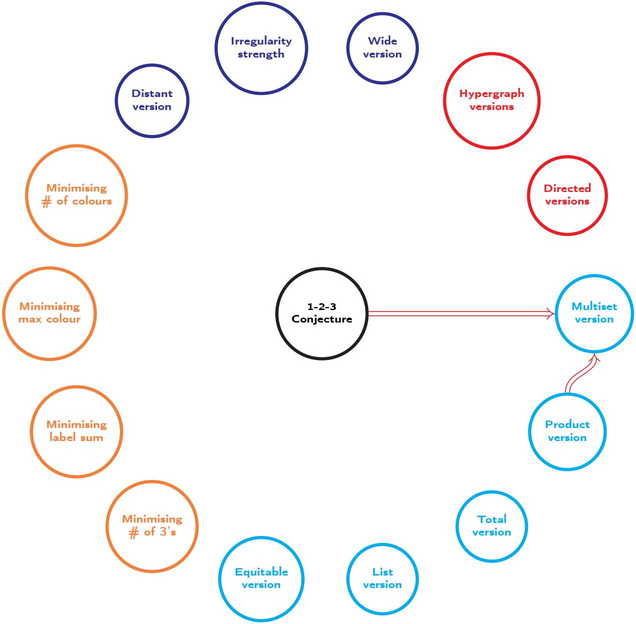

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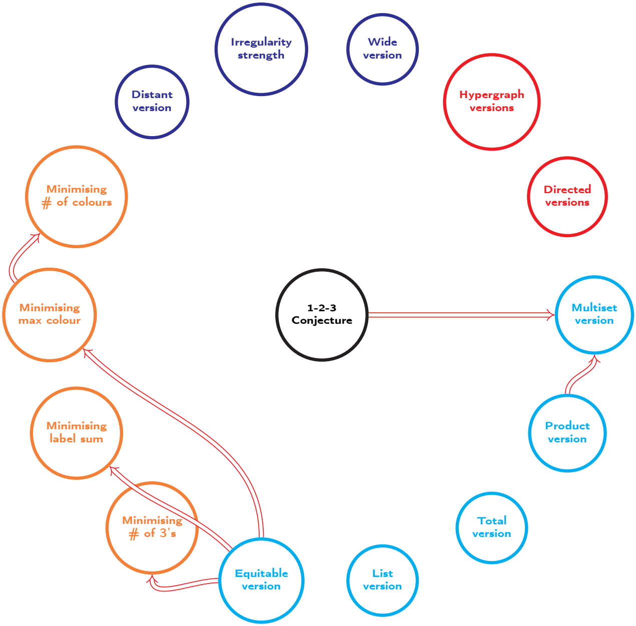

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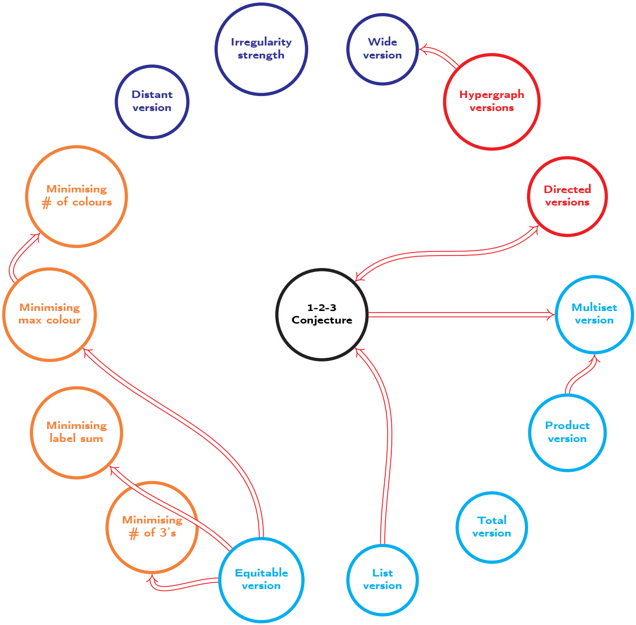

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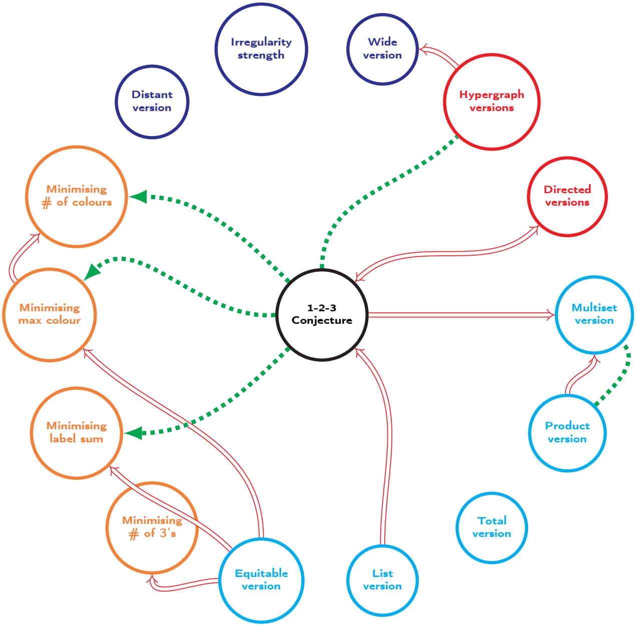

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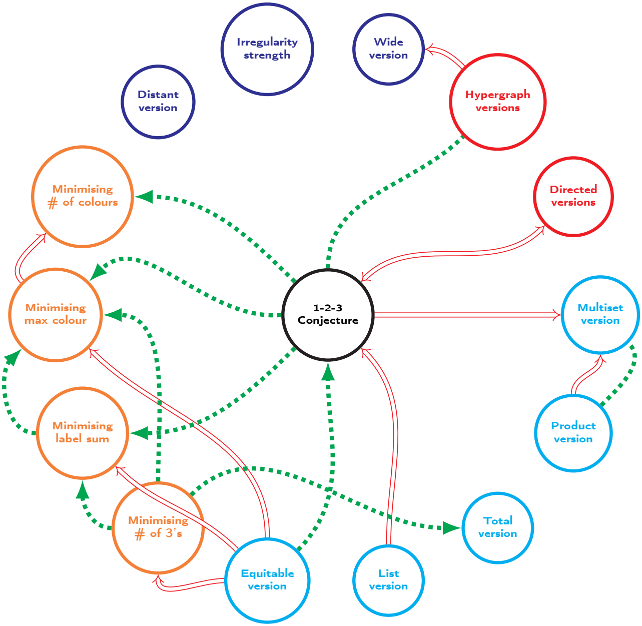

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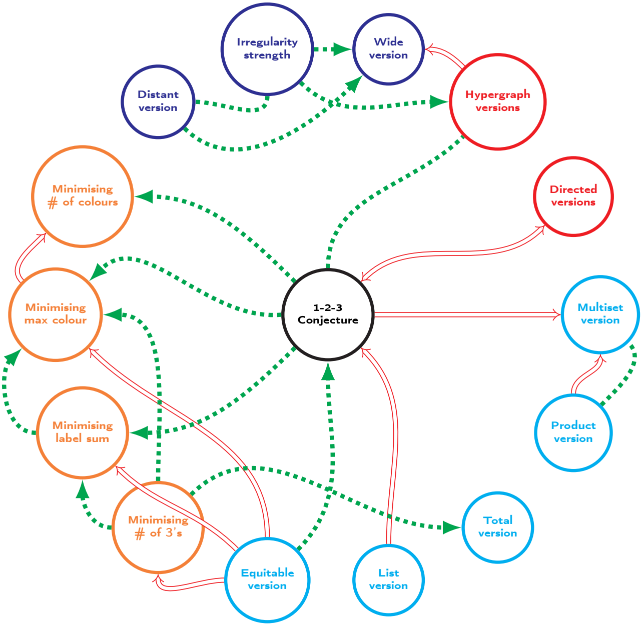

Conclusion and perspectives

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Thank you for your attention!