Nowhere-zero flows and perfect matchings

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Journée Francilienne de Recherche Opérationnelle

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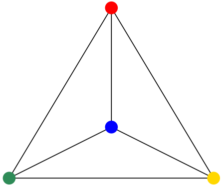
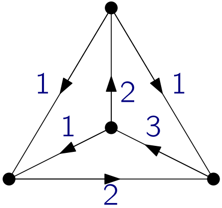
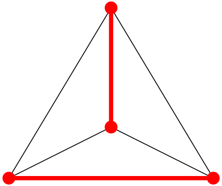
Colorings, flows, and perfect matchings







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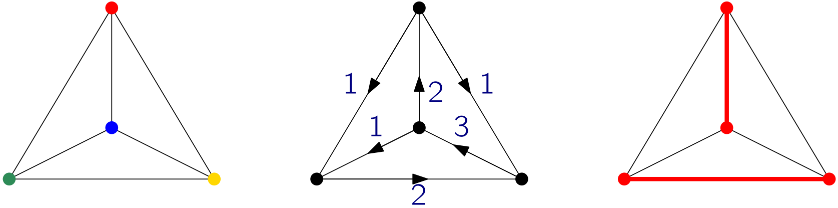

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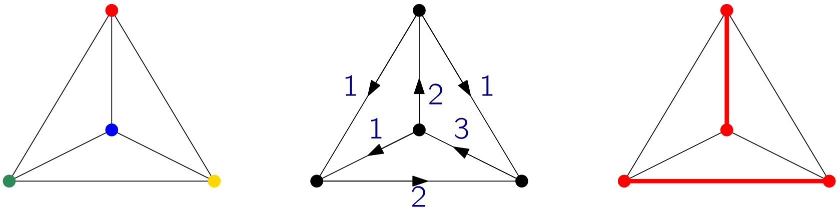

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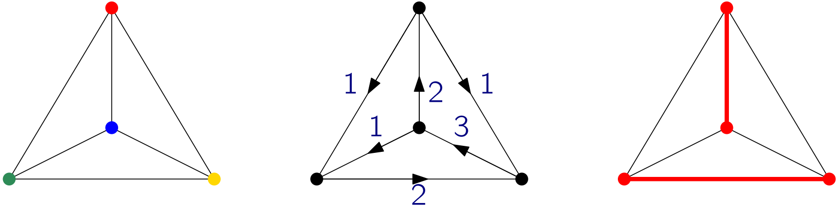

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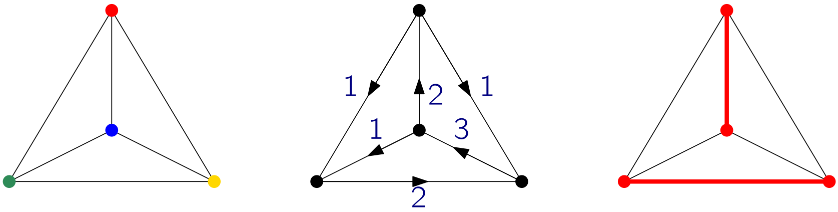

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The Four color theorem

Planar graphs are 4-colorable.

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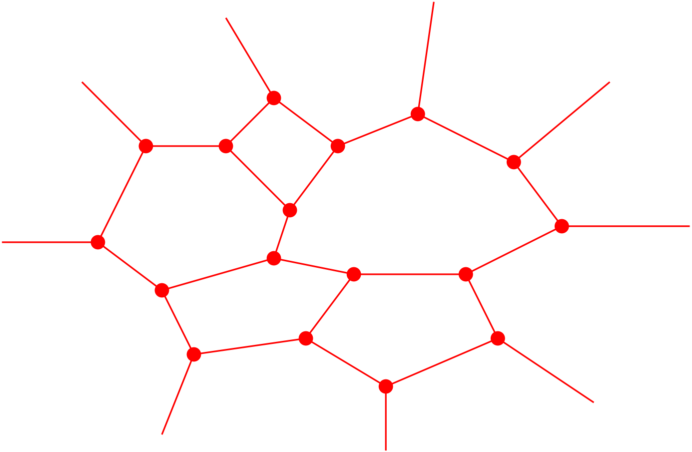

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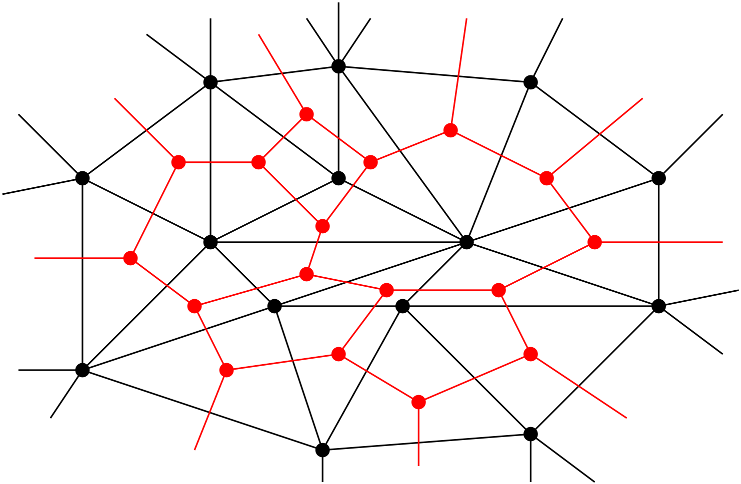

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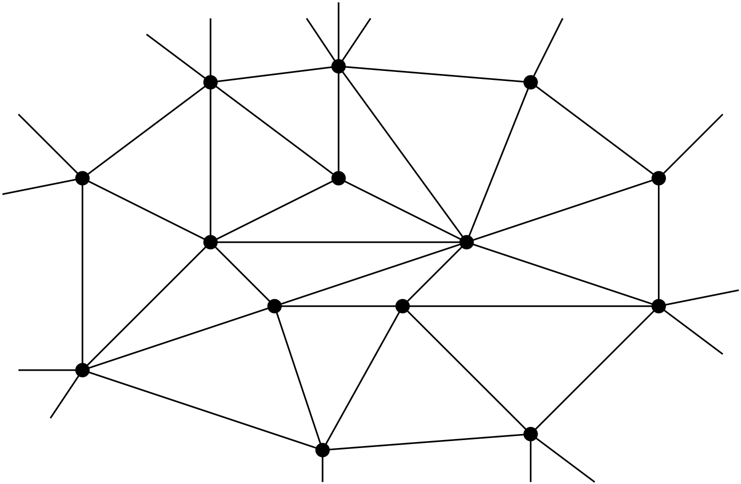

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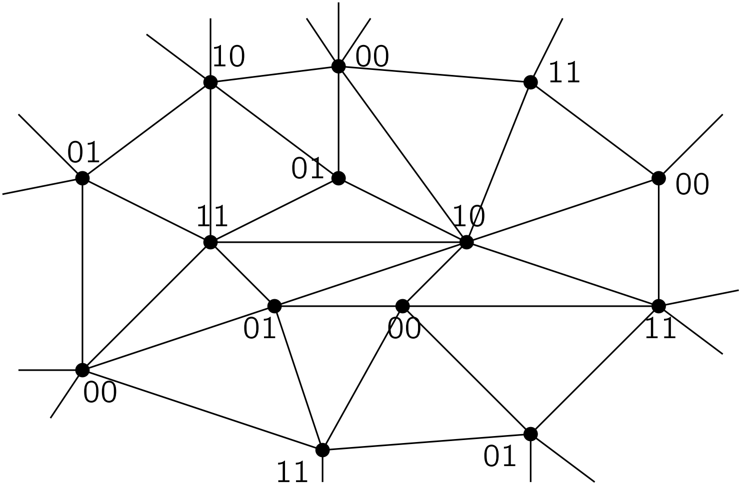

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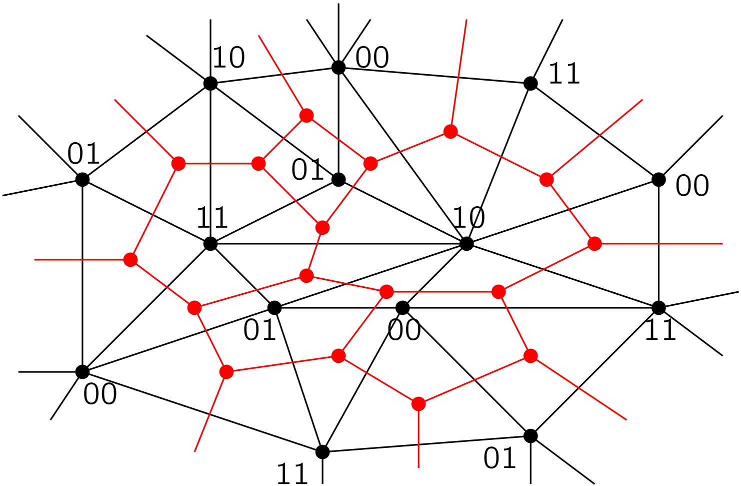

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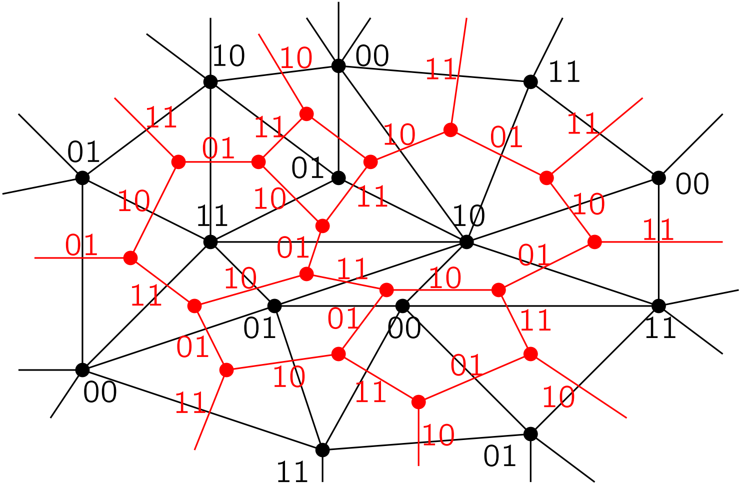

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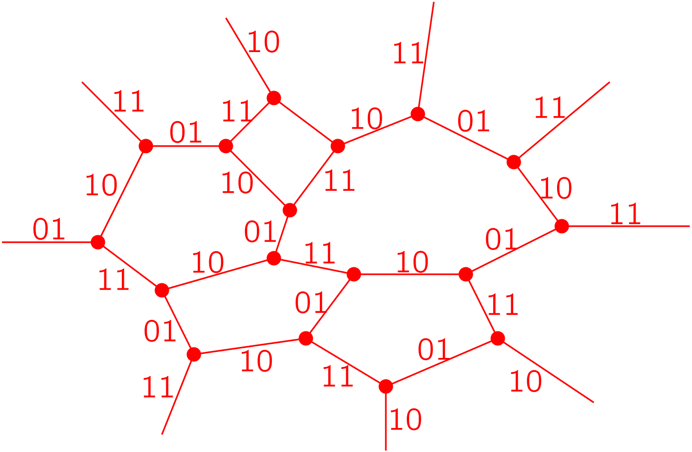
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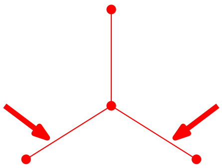

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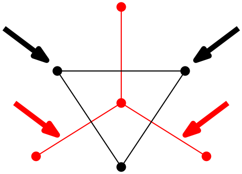

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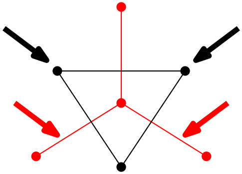

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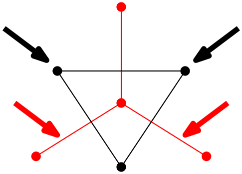

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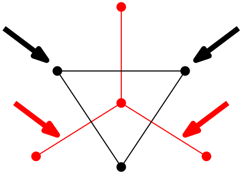

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Every 2-edge-connected cubic planar graph has a nowhere-zero 4-flow.

Flows in graphs

A k-flow in an oriented graph G is an assignment of integers of $\pm\{0, 1, \ldots, k-1\}$ to the arcs of G, such that for each vertex v, the sum of the values of the arcs entering v is equal to the sum of the values of the arcs leaving v.

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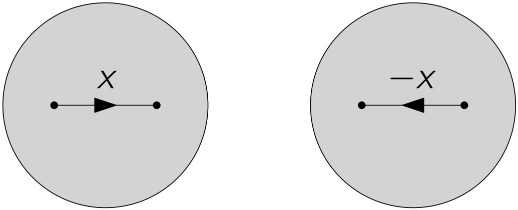

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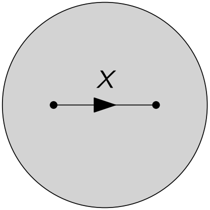

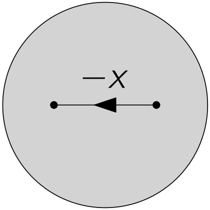

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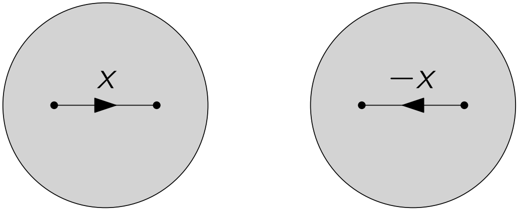

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For plane graphs, flows are in duality with proper vertex colorings: G has a k-coloring if and only if its dual G^* has a nowhere-zero k-flow.

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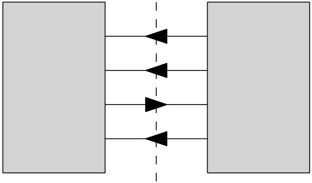

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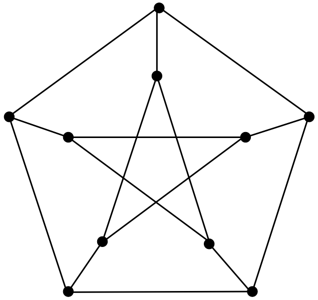

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For instance: any 2-edge-connected graph has a 4-flow with at most $\frac{1}{15}$ of its edges with flow value zero.

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Theorem (Lovász, Thomassen, Wu, Zhang 2013)

Every 6-edge-connected graph has a nowhere-zero 3-flow.

Cubic graphs and perfect matchings

Reminder: cubic graphs have nowhere-zero 4-flows if and only of they are 3-edge-colorable, i.e. their edge-set can be covered by 3 perfect matchings.

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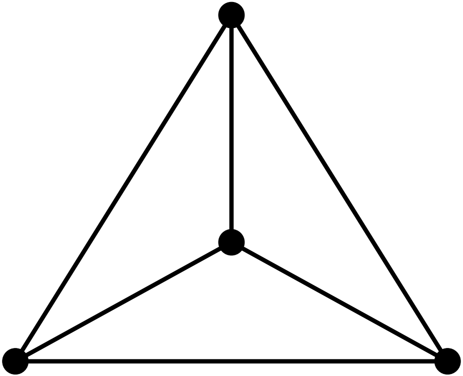

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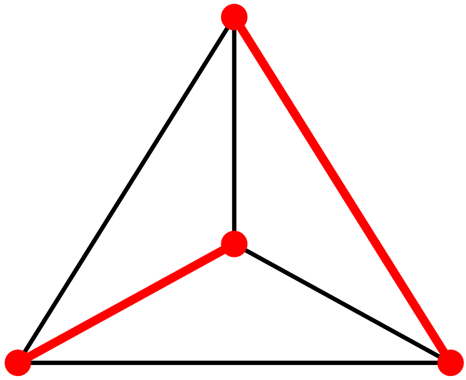

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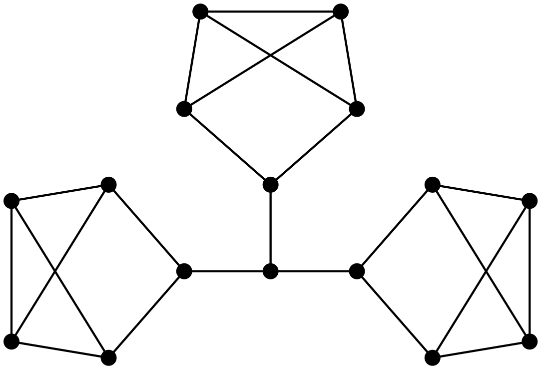

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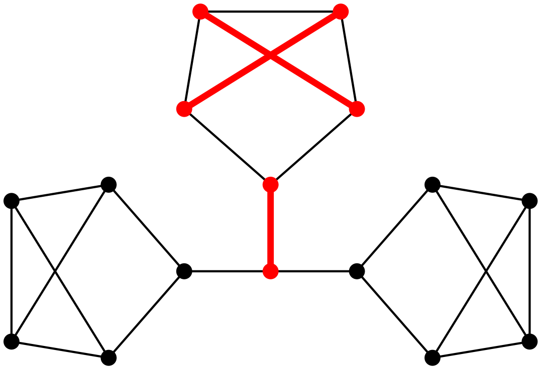

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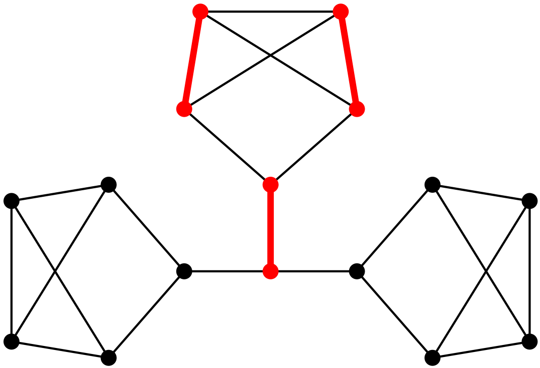

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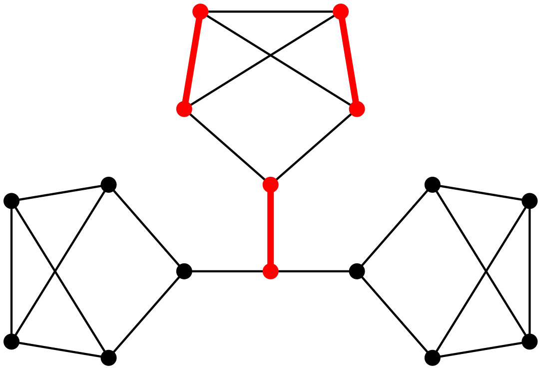

Theorem (Petersen 1891)

Every cubic 2-edge-connected graph contains a perfect matching.

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Theorem (Edmonds 1965)

A vector $w \in \mathbb{R}^E$ is in the perfect matching polytope if and only if (i) for each edge e, $w_e \geq 0$, (ii) for each vertex v, $\sum_{e \ni v} w_e = 1$, and (iii) for each odd edge-cut C, $\sum_{e \in C} w_e \geq 1$.

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Conjecture (Berge, Fulkerson 1971)

For any cubic 2-edge-connected graph G, the vector $\frac{1}{3}$ can be expressed as a convex combination of at most 6 perfect matchings of G.

Berge-Fulkerson conjecture

Equivalently:

Conjecture (Berge, Fulkerson 1971)

Every cubic 2-edge-connected graph contains six perfect matchings (with repetitions allowed) covering each edge precisely twice.

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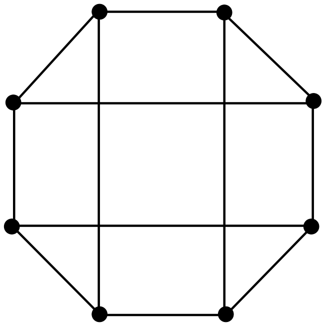

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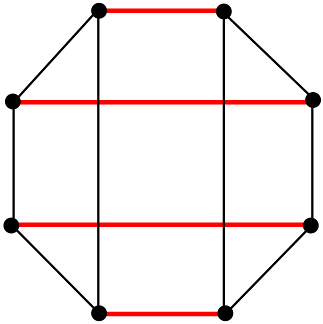

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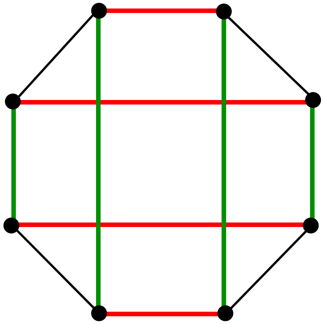

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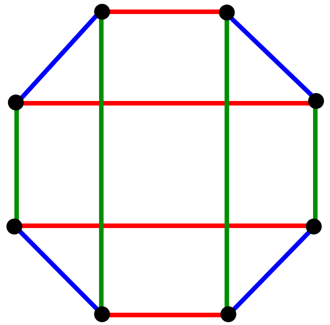

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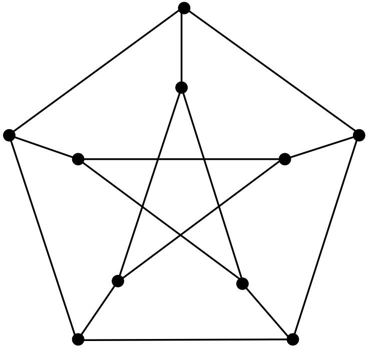

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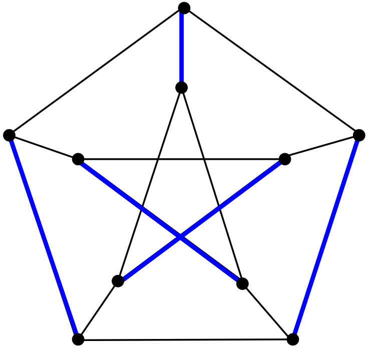

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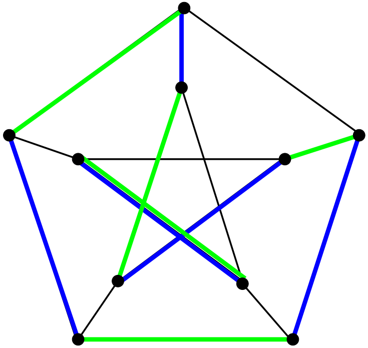

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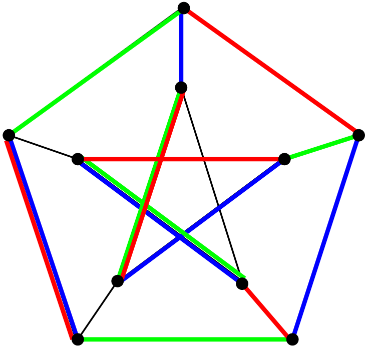

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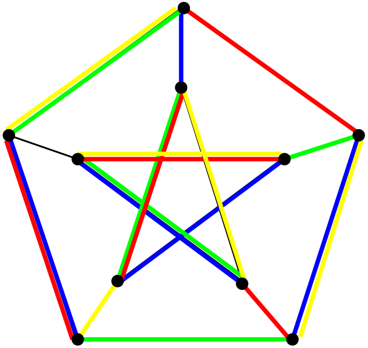

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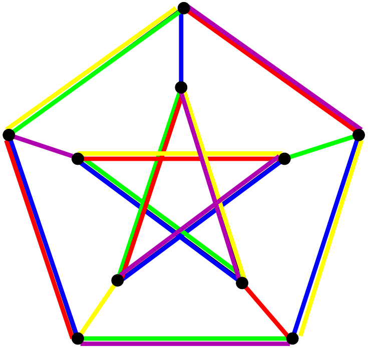

Berge-Fulkerson conjecture

Equivalently:

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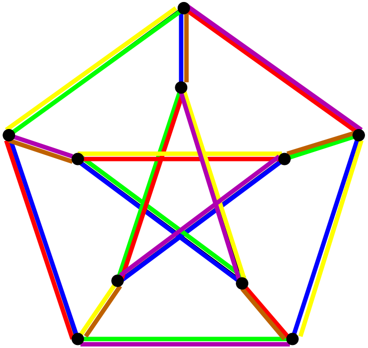

Covering the edge-set with perfect matchings

Berge-Fulkerson conjecture would imply

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The edge-set of every cubic 2-edge-connected graph can be covered by at most 5 perfect matchings.

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To achieve $\log n$: Draw random perfect matchings from the $\frac{1}{3}$-distribution until all edges are covered.

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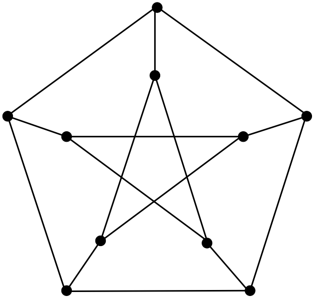

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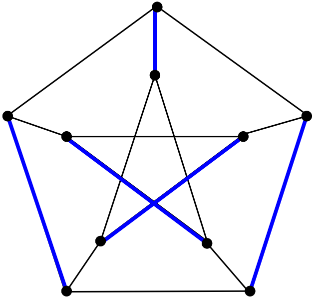

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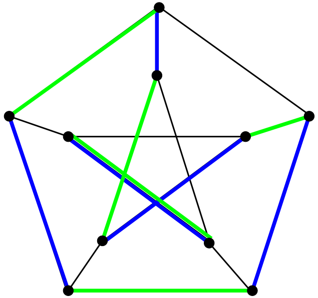

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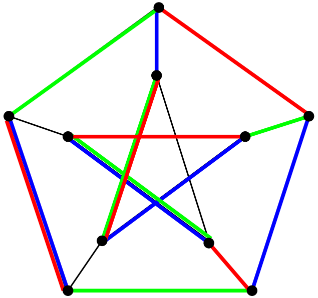

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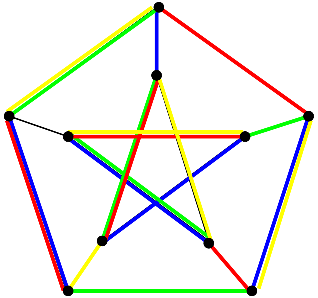

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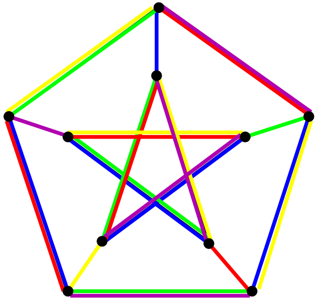

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Berge-Fulkerson Conjecture implies $m_3 = \frac{4}{5}$ and $m_4 = \frac{14}{15}$

Graphs G with $m_4(G) = 1$

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Deciding whether $m_4(G) = 1$ for a cubic 2-edge-connected graph G is an NP-complete problem.

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Fan-Raspaud Conjecture and $m_3 = 4/5$

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Question: Does Fan-Raspaud Conjecture imply $m_3 = \frac{4}{5}$?

More open problems

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3. Is there a constant c such that any cubic 2-edge-connected graph has at most c perfect matchings with empty intersection.